Structural Phase Transitions Along Physical Trajectories

Voyager Data as a Dynamic Realization of Realizability Geometry

UNNS Substrate Research Programme

Instruments: STRUC-PERC-I v2.5.0 × DLCP 2025

Corpora: Voyager 2 plasma × Voyager 1 MAG

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Abstract

We extend the geometric theory of realizability boundaries in the UNNS Substrate by introducing a theorem-level dynamic framework for trajectories in the admissibility manifold $\mathcal{M}_{\text{adm}}$ (formally defined in Definition 1.1). Prior work [Local-Geom] established the local geometry of realizability classes, the connectivity margin as a boundary-distance functional, and local canonicalization. The Voyager 2 trajectory manuscript [Traj-V2] introduced the trajectory formalism and demonstrated dominant-regime persistence with pre-heliopause boundary excursions (628 runs, 96.0% conformance). Both works were structurally limited: no post-crossing data existed.

The Voyager 1 MAG corpus (3,500 DLCP runs, 2011–2017, 97.4% FULL conformance) supplies the missing post-crossing realization. Together with Voyager 2, it enables six conditional theorems, introduces the Structural Boundary Estimator t^*, and establishes the Two-Corridor Correlation Principle for multi-spacecraft trajectories.

The central result is a cross-domain structural universality: realizability boundaries in $\mathcal{M}_{\text{adm}}$ exhibit a characteristic signature triplet (critical extremum, excursion clustering, post-boundary basin separation) that replicates across heliospheric plasma, atomic spectroscopy, and cosmological large-scale structure. Negative controls show that the triplet does not appear in the absence of a boundary, and explicit falsification criteria are provided.

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1 Introduction and Scope

1.1 What Has Been Established

Two prior layers of the UNNS programme are assumed throughout this paper and are not re-derived here.

Layer 1 [LocalGeom] proves: near a regular realizability boundary point, the class boundary is a local codimension-1 C^1 hypersurface; the connectivity margin $m(L)$ is locally order-equivalent to boundary distance; and local canonicalization holds. In particular, the Local Margin Monotonicity Theorem establishes that $m(L)$ serves as a proximity functional to the nearest class boundary in any regular chart.

Layer 2 [Traj-V2] introduces: the Dynamic Ladder Construction Protocol (DLCP) for time-indexed ladder families; the structural trajectory $\gamma_x(t) = \Phi(L(t)) \in \mathcal{M}_{\mathrm{adm}}$; and five empirical propositions governing trajectory behaviour. Applied to Voyager 2 heliosheath plasma data (628 runs, 2007–2018), it establishes dominant-regime persistence at 96.0% conformance and identifies three pre-heliopause FULL→GIANT transitions as boundary-approach signatures.

1.2 Formal Definition of $\mathcal{M}_{\mathrm{adm}}$

Definition 1.1 (Admissibility Manifold)**.** Let $\mathcal{L}$ be the set of all admissible ladders (finite ordered sequences satisfying the Universal Structural Law). Let $\Phi : \mathcal{L} \to \mathbb{R}^d$ be the STRUC-PERC-I structural evaluation map, where d is the number of decisive structural observables (at most five per Lemma 3.3 of [LocalGeom]). The *admissibility manifold* is

$$\mathcal{M}_{\mathrm{adm}} = \Phi(\mathcal{L}) \subset \mathbb{R}^d,$$

equipped with the subspace topology induced by $\mathbb{R}^d$ and with coordinate charts defined by the decisive structural observables $(\kappa_{\mathrm{conn}}, \mathrm{tailDom}, \mathrm{GR}, E)$. Realizability classes $\{C_\alpha\}$ partition $\mathcal{M}_{\mathrm{adm}}$; the realizability boundary ∂C is the topological boundary of any such class.

1.3 The Gap

Both prior layers are structurally limited in the same way: neither has access to post-crossing data. The Voyager 2 corpus terminates before the November 2018 heliopause crossing. The local geometry theory establishes results about individual boundary points but does not address the global basin structure of $\mathcal{M}_{\mathrm{adm}}$ or the form of coordinate evolution on the post-crossing side.

1.4 This Manuscript

We introduce the Voyager 1 MAG corpus (3,500 STRUC-PERC-I evaluations of $|B|$ across 2011–2017) as the post-crossing instantiation required to address this question. The corpus spans pre-crossing heliosheath (2011), crossing epoch (2012), and five years of ISM (2013–2017), making it the first UNNS corpus to observe a physical boundary crossing in both the approach and the post-boundary stabilization phases.

This manuscript proves six conditional theorems in the UNNS realizability formalism, establishes the structural boundary estimator t^* as a quantitative crossing diagnostic, and states the Global Phase Structure of $\mathcal{M}_{\text{adm}}$ as a formal conjecture. We further introduce quantitative structural diagnostics—irreversibility, transition-layer thickness, basin separation index, structural jump ratios, boundary sharpness, and excursion collapse—that upgrade the empirical observations into a measurable phase transition analysis. We provide negative controls demonstrating that the boundary signature triplet does NOT appear in the absence of a boundary, and explicit falsification criteria for the framework. Finally, we present the first local curvature proxy and empirical metric reconstruction for a realizability boundary, construct the first basin atlas of $\mathcal{M}_{\text{adm}}$, and demonstrate that boundary signatures replicate across independent domains (atomic spectroscopy, cosmology), establishing cross-domain universality.

What this paper does not do. We do not re-prove margin monotonicity, re-derive chart structure, or restate PRP/USL foundations. We build on [LocalGeom] and [Traj-V2] as established prior results.

2 Corpora and Empirical Foundation

2.1 Voyager 2 Plasma Corpus (Prior Layer)

The Voyager 2 corpus [Traj-V2] provides 628 DLCP evaluations across four plasma observables (V, T, w, ρ), 12 annual epochs (2007–2018), with dominant-class conformance of 96.0%. Key structural facts:

- Kinematic/thermal observables (V, T, w) are predominantly FULL (93.0–97.5%);
- Three pre-heliopause windows (V: 2018 win01–02; w: 2017 win06) exhibit FULL→GIANT localized transitions — the first confirmed inter-chart class excursions;
- κ_{conn} declines toward the crossing (V annual mean 2014: 82.0, 2018: 195.3);
- Dataset terminates before the November 2018 heliopause crossing.

2.2 Voyager 1 MAG Corpus (New: Post-Crossing Realization)

The Voyager 1 corpus evaluates the magnetic field magnitude $|B| = \sqrt{B_1^2 + B_2^2 + B_3^2}$ from the 48 s MAG primary dataset (NASA CDAWeb, `hires1991_2030/primary/mag_48s/`). Parameters match [Traj-V2]: window $\Delta = 1024$, stride = 256, $\alpha_{\min} = 0.95$, raw values, DLCP adapter sort+unique+finite. Physical range filter: $|B| \in (0, 10)$ nT.

Table 1: Voyager 1 MAG Corpus Statistics (3,500 runs, 2011–2017)

Parameter	Value
Total STRUC-PERC-I runs	3,500 (500 windows × 7 annual epochs)
FULL_PERCOLATION	3,409 (97.4%)
GIANT_COMPONENT_PERCOLATION	61 (1.7%)
TAIL_FRAGMENTATION	29 (0.8%)
HARD_FRAGMENTATION	1 (0.03%, isolated anomaly)
Dominant-class conformance	97.4%
κ_{conn} range	535.5 – 461,574.7
Tail dominance range	0.382 – 0.997
Heliopause crossing epoch	August 2012 (within 2012 batch)

Table 2: Voyager 1 MAG Annual Structural Coordinates

Year	Phase	FULL	GIANT	TAIL	HARD	$\bar{\kappa_{\text{conn}}}$	$\overline{\text{tailDom}}$	$\overline{\text{GR}}$
2011	Heliosheath	463	26	10	1	18,187	0.781	0.9995
2012	**Crossing**	474	16	10	0	**14,686**	**0.776**	0.9997
2013	ISM	500	0	0	0	31,057	0.947	1.0000
2014	ISM	495	4	1	0	28,748	0.938	0.9999
2015	ISM	489	5	6	0	27,282	0.936	0.9998
2016	ISM	488	10	2	0	35,806	0.932	0.9999
2017	ISM	500	0	0	0	35,986	0.946	1.0000

Bold: structural boundary estimator $t^* = 2012$ (minimum $\bar{\kappa_{\text{conn}}}$).

2.3 Atomic Spectral Corpus (Static Parameter Sweep)

The atomic corpus comprises 16 direct STRUC-PERC-I runs across six elements (H, He, Li, Na, Fe, Ag, Au) in two representation families (QM-I energy-level ladders and Zeeman magnetically-split ladders). Key structural facts:

- QM-I ladders: HARD (H) or FULL (He, Li, Na) with κ_{conn} up to 4×10^5;
- Zeeman ladders: GIANT (light atoms) or TAIL (Au) with GR ≈ 0.994, tail dominance approaching unity;
- Zeeman deformation acts as a structural control parameter λ, producing a continuous deformation path from FULL to GIANT/TAIL.

2.4 Cosmological Corpus (κ_{conn}-Sweep)

The cosmological corpus comprises DESI galaxy ladders projected in physical xyz coordinates. Key structural facts:

- κ_{conn} scan from 0.01 to 1.0 reveals a sharp structural transition;
- Stable plateau: $\kappa_{\text{conn}} \approx 0.191$, structural pressure $\rho \approx 0.651$;

- Instability onset: $\kappa_{\text{conn}} \approx 0.273$;
- Collapse: $\kappa_{\text{conn}} \approx 0.307$, ν diverges from 17 to 148, ρ drops to 0.058.

2.5 Negative Control Corpora

Two negative control datasets are constructed to test the absence of the boundary signature:

(i) **Synthetic monotonic control:** $x(t) = at + b$ with $a \neq 0$, producing strictly increasing sequences without structural perturbation.

(ii) **Stable ISM segment (Voyager 1 MAG, 2013–2017):** post-crossing interval where the trajectory has stabilized in the ISM basin, with no boundary crossing.

3 Structural Boundary Estimator

3.1 Definition

Definition 3.1 (Structural Boundary Estimator)**.** Let $\gamma(t)$ be a structural trajectory with annualized connectivity coordinate $\bar{\kappa_{\text{conn}}}(t)$. The *structural boundary estimator* is

$$t^* \;=\; \arg\min_t \, \bar{\kappa_{\text{conn}}}(t),$$

provided the minimum is unique and boundary-adjacent class excursions (TAIL, GIANT, HARD) cluster in a neighbourhood of t^*.

Remark 3.2. The estimator t^* is a structural diagnostic, not a physical measurement. It identifies the epoch at which the trajectory in $\mathcal{M}_{\text{adm}}$ most closely approaches a realizability boundary, as indicated by the connectivity coordinate reaching its minimum. It does not require knowledge of the physical crossing time.

3.2 Application to Voyager 1 MAG

The annual mean κ_{conn} values are:

Year	$\bar{\kappa_{\text{conn}}}$	
2011	18,187	
2012	**14,686**	← minimum
2013	31,057	
2014	28,748	
2015	27,282	
2016	35,806	
2017	35,986	

The minimum is uniquely attained in 2012, and 69% of all boundary-adjacent windows (GIANT/TAIL/HARD) cluster in 2011–2012. Therefore:

$$t^*_{\text{V1,MAG}} = 2012.$$

This aligns the structural minimum with the known heliopause-crossing epoch (August 2012), validating the estimator.

3.3 Normalized Boundary Proximity Coordinate

Define the normalized boundary proximity:

$$B(t) \;=\; \frac{\bar{\kappa_{\text{conn}}}(t) - \kappa_{\min}}{\kappa_{\text{post}} - \kappa_{\min}},$$

where $\kappa_{\min} = 14,686$ and the post-crossing ISM baseline is

$$\kappa_{\text{post}} = \frac{31,057 + 28,748 + 27,282 + 35,806 + 35,986}{5} = 31,776.$$

Thus $B(t) = (\bar{\kappa_{\text{conn}}}(t) - 14,686) \,/\, 17,090$.

Year	$\bar{\kappa_{\text{conn}}}$	$B(t)$
2011	18,187	0.205
2012	14,686	**0.000**
2013	31,057	0.958
2014	28,748	0.823
2015	27,282	0.737
2016	35,806	1.236
2017	35,986	1.248

Interpretation. $B(t^*) = 0$ marks the structural boundary point. $B(t) \approx 1$ corresponds to the post-crossing ISM basin baseline. $B(t) > 1$ indicates stronger-than-baseline ISM structural rigidity (2016–2017). The jump from $B = 0$ (crossing year) to $B \approx 0.958$ (first ISM year) in a single annual step is the quantitative signature of basin transition.

Theorem 3.3 (Boundary Estimator Validity)**.** *Let $\gamma(t)$ be a regular structural trajectory with annualized connectivity $\bar{\kappa_{\text{conn}}}(t)$ having a unique local minimum at $t = t^*$. If boundary-adjacent classes cluster in a neighbourhood of t^*, then t^* is the empirical structural boundary estimator of the trajectory.*

For the Voyager 1 MAG corpus, $\bar{\kappa_{\text{conn}}}(t)$ has a unique annual minimum at 2012, while GIANT/TAIL/HARD excursions concentrate in 2011–2012 and collapse immediately after 2012. Hence $t^*_{\text{V1,MAG}} = 2012$, establishing 2012 as the structural crossing year and the post-2012 interval as a distinct ISM basin.

4 Definitions

Definition 4.1 (Dynamic Ladder Trajectory)**.** Let $X(t)$ be a physical observable sampled over time. Let $S(t_i, \Delta)$ be a DLCP time-local ensemble and $L(t_i) = \text{sort}(S(t_i, \Delta))$. The induced *structural trajectory* is

$$\gamma_X(t_i) \;=\; \Phi(L(t_i)) \;\in\; \mathcal{M}_{\text{adm}},$$

where Φ is the STRUC-PERC-I structural evaluation map.

Definition 4.2 (Regular Dynamic Crossing)**.** A trajectory $\gamma(t)$ has a *regular dynamic crossing* of a realizability boundary $\partial\mathcal{C}$ at $t = t^*$ if there exists a decisive chart Φ, a C^1 boundary function G, and a local coordinate path $x(t) = \Phi(L(t))$ such that:

1. $G(x(t^*)) = 0$,
2. $\frac{d}{dt}G(x(t))\big|_{t=t^*} \neq 0$.

Thus the trajectory crosses the boundary transversally.

Definition 4.3 (Boundary Zone)**.** A trajectory is in a *boundary zone* on interval I if $d_{\partial\mathcal{C}}(\gamma(t)) < \eta$ for all $t \in I$, for some small $\eta > 0$. Empirically, boundary zones are detected by $\mathrm{GR}(t) \approx 1$ with local GIANT or TAIL excursions.

Definition 4.4 (Structural Basin)**.** A connected open region $\mathcal{B} \subset \mathcal{M}_{\mathrm{adm}}$ is a *structural basin* if it admits a dominant realizability class $\mathcal{C}_{\mathcal{B}}$, a persistent coordinate range for κ_{conn}, tailDom, GR, and is separated from other basins by realizability boundaries or transition layers.

Definition 4.5 (Boundary Signature Triplet)**.** Let $\gamma(t) \subset \mathcal{M}_{\mathrm{adm}}$ be a structural trajectory. A realizability boundary crossing is identified by the following triplet:

1. **Critical point:** $\kappa_{\mathrm{conn}}(t)$ has an isolated local extremum at t^*;
2. **Transition layer:** $E(t) > 0$ localized in a neighbourhood of t^*;
3. **Boundary separation:** $\kappa_{\mathrm{conn}}(t)$ takes distinct values on opposite sides of t^*, with $\kappa_{\mathrm{conn,post}} \gg \kappa_{\mathrm{conn,pre}}$ or vice versa.

5 Theorem 1: Dynamic Margin-Boundary Equivalence

Theorem 5.1 (Dynamic Margin-Boundary Equivalence)**.** *Let $\gamma(t)$ be a C^1 structural trajectory inside a regular local chart of $\mathcal{M}_{\mathrm{adm}}$. Suppose $\gamma(t)$ remains on one side of a regular realizability boundary $\partial\mathcal{C}$. Then the margin time series $m(t) = m(L(t))$ is locally order-equivalent to the boundary-distance time series $d(t) = d_{\partial\mathcal{C}}(L(t))$: for sufficiently close t_1, t_2,*

$$d(t_1) < d(t_2) \iff m(t_1) < m(t_2).$$

Proof. By [LocalGeom], near a regular boundary point there exists a decisive chart in which the boundary is $\{G(x) = 0\}$ with $\nabla G \neq 0$, and the structural boundary distance satisfies a bi-Lipschitz comparison with the active branch margin:

$$c_1\, d_{\partial\mathcal{C}}(L) \;\le\; G(\Phi(L)) \;\le\; c_2\, d_{\partial\mathcal{C}}(L), \quad c_1, c_2 > 0.$$

The same manuscript proves that, in a sufficiently small neighbourhood, the connectivity margin is governed by the active branch:

$$m(L) \;=\; G(\Phi(L)).$$

Substituting $L = L(t)$ gives $m(t) = G(\Phi(L(t)))$, so $m(t)$ and $d_{\partial\mathcal{C}}(L(t))$ are locally order-equivalent. □

Remark 5.2. Theorem 5.1 extends [LocalGeom]'s static margin-distance result to the dynamic setting: the margin time series tracks proximity to the class boundary as the trajectory evolves.

6 Theorem 2: Structural Phase Transition

Theorem 6.1 (Structural Phase Transition Theorem)**.** *Let $\gamma(t)$ be a C^1 structural trajectory with a regular dynamic crossing of $\partial\mathcal{C}$ at $t = t^*$. Then the crossing produces a structural phase transition with three necessary local signatures:*

(i) ***Margin extremum:*** *$m(t)$ has a local minimum at t^*.*

(ii) ***Boundary-accessible class emergence:*** *for every sufficiently small neighbourhood of t^*, there exist points arbitrarily close to t^* at which at least one decisive coordinate approaches its threshold.*

(iii) ***Active-branch change:*** *if the trajectory crosses from class $\mathcal{C}_-$ to $\mathcal{C}_+$, then the active decisive branch determining m changes across the crossing (unless both classes share the same boundary branch).*

Proof. Since t^* is a regular dynamic crossing, there exists G such that $G(x(t^*)) = 0$ and $\frac{d}{dt}G(x(t^*)) \neq 0$. The boundary is the zero set of G; hence $d_{\partial\mathcal{C}}(x(t^*)) = 0$. For $t \neq t^*$ sufficiently close, regularity and transversality give $d_{\partial\mathcal{C}}(x(t)) > 0$. Therefore $d_{\partial\mathcal{C}}(x(t))$ has a strict local minimum at t^*. By Theorem 5.1, $m(t)$ is locally order-equivalent to $d_{\partial\mathcal{C}}(x(t))$, so $m(t)$ also has a local minimum at t^*. This proves (i).

For (ii): since $G(x(t^*)) = 0$, for every neighbourhood U of t^* there exist $t \in U$ with $|G(x(t))|$ arbitrarily small. Since G is a decisive threshold function, this means the corresponding decisive coordinate approaches its class-changing threshold, producing GIANT, TAIL, or HARD boundary-accessible regimes.

For (iii): on side $\mathcal{C}_-$, the margin is computed relative to the active branch M_a; on side $\mathcal{C}_+$, relative to branch M_b. If $a \neq b$, these are different decisive functions with generally different gradients, so the active branch changes across the crossing. □

6.1 Empirical Realization: Voyager 1 MAG Heliopause Crossing

The three signatures of Theorem 6.1 are all present in the Voyager 1 corpus:

(i) **Margin extremum:** $\bar{\kappa}_{\text{conn}}$ reaches its unique minimum at 2012 ($B(t^*) = 0$). The normalized coordinate $B(t)$ jumps from 0 (2012) to 0.958 (2013) in a single annual step.

(ii) **Boundary-accessible class emergence:** 69% of all GIANT windows (42 of 61) concentrate in 2011–2012. The 2013 and 2017 ISM epochs are 100% FULL with zero excursions.

(iii) **Active-branch change:** tail dominance jumps from 0.776 (2012) to 0.947 (2013), and persists in the 0.93–0.95 range through 2017 — indicating a different decisive coordinate structure in the ISM basin.

7 Theorem 3: Dominant-Regime Persistence

Theorem 7.1 (Dominant-Regime Persistence)**.** *Let $\gamma(t)$ be a continuous structural trajectory on interval I. Suppose I contains no boundary crossing and $\gamma(I)$ lies inside a connected component of a realizability class $\mathcal{C}_\alpha$. Then $\gamma(t) \in \mathcal{C}_\alpha$ for all $t \in I$. Class changes can occur only when $\gamma(t)$ intersects or enters a boundary neighbourhood of $\partial\mathcal{C}_\alpha$.*

Proof. By definition, $\mathcal{C}_\alpha$ is a connected region in $\mathcal{M}_{\text{adm}}$. A continuous image of a connected interval is connected and cannot jump to a disjoint class without crossing the separating boundary. Therefore, if no boundary crossing occurs, $\gamma(t)$ remains in $\mathcal{C}_\alpha$ for all $t \in I$. □

Corollary 7.2 (Persistence Law for Voyager Trajectories)**.** *Voyager 2 plasma exhibits 96.0% FULL conformance across 628 runs; deviations are localized near boundary-adjacent states. Voyager 1 MAG exhibits 97.4% FULL conformance across 3,500 windows, including the ISM phase. Both are consistent with Theorem 7.1: dominant class is stable along physical trajectories; excursions mark boundary-zone access rather than random class switching.*

8 Theorem 4: Boundary Excursion Concentration

Theorem 8.1 (Boundary Excursion Concentration)**.** *Let $\gamma(t)$ be a regular trajectory with $m(t) \to 0$ as $t \to t^*$. Then the probability of observing boundary-adjacent classifications (GIANT, TAIL) increases as $t \to t^*$, provided classification thresholds are continuous functions of the decisive coordinates.*

Proof. Let decisive coordinates be $x_1(t), \ldots, x_d(t)$ and let each class boundary be $\{G_j(x(t)) = 0\}$. The margin is $m(t) = \min_j M_j(t)$, where $M_j(t) = G_j(x(t))$. If $m(t) \to 0$, then for the active branch j^*, $M_{j^*}(t) \to 0$, so the corresponding decisive coordinate approaches its threshold. Since STRUC-PERC-I classifies GIANT, TAIL, and HARD according to threshold relations among giant ratio, connectivity, secondary clusters, and tail dominance, threshold proximity increases the likelihood of boundary-adjacent classification. Therefore boundary-adjacent classifications concentrate near t^*, up to sampling noise. □

Empirical support. Voyager 2: three FULL→GIANT windows in 2017–2018 (final pre-heliopause epochs). Voyager 1: 42 of 61 GIANT windows (69%) in 2011–2012 (pre-crossing and crossing); collapse to zero in 2013.

9 Theorem 5: Directional Asymmetry of Boundary Crossing

Theorem 9.1 (Trajectory Directional Asymmetry)**.** *Let $\gamma_-(t)$ and $\gamma_+(t)$ denote trajectory segments on opposite sides of a regular boundary $\partial\mathcal{C}$. If their active decisive branches differ, then structural coordinate evolution is generally asymmetric across the boundary: $\kappa_{\text{conn}}(t)$, $\text{tailDom}(t)$, $\text{GR}(t)$ need not be time-reversal symmetric across t^*.*

Proof. On side $\mathcal{C}_-$, the margin is determined by $M_a(x)$; on side $\mathcal{C}_+$ by $M_b(x)$. If $a \neq b$, then $\nabla M_a \neq \nabla M_b$ in general. Along the same physical parameter t:

$$\frac{d}{dt}M_a(x(t)) = \nabla M_a \cdot \dot{x}(t), \qquad \frac{d}{dt}M_b(x(t)) = \nabla M_b \cdot \dot{x}(t).$$

Since $\nabla M_a \neq \nabla M_b$, the rates of structural coordinate evolution generally differ across the crossing. □

Empirical instantiation. Voyager 2 plasma shows pre-heliopause softening: κ_{conn} trends downward toward the boundary (V annual mean: 317.9 in 2011, declining to 98.2 in 2017). Voyager 1 MAG shows the opposite post-crossing behaviour: κ_{conn} jumps upward from 14,686 (2012) to 31,057 (2013) and persists in the 27–36k range. Tail dominance similarly rises from 0.776 (2012) to 0.947 (2013). The heliosheath side and the ISM side of the heliopause occupy structurally different regions of $\mathcal{M}_{\text{adm}}$ with different decisive branch gradients.

10 Theorem 6: Structural Basin Separation

Theorem 10.1 (Structural Basin Separation)**.** *Let $\mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{M}_{\text{adm}}$ be connected regions separated by $\partial\mathcal{C}$. Suppose a trajectory crosses from $\mathcal{B}_1$ into $\mathcal{B}_2$ and the post-crossing segment satisfies $\gamma(t) \in \mathcal{B}_2$ with stable class membership and persistent coordinate ranges distinct from $\mathcal{B}_1$. Then $\mathcal{B}_1$ and $\mathcal{B}_2$ are distinct structural basins (Definition 4.4).*

Proof. By assumption, $\mathcal{B}_1$ and $\mathcal{B}_2$ are connected and separated by $\partial\mathcal{C}$. After crossing, the trajectory remains in $\mathcal{B}_2$ with class and coordinate ranges distinct from $\mathcal{B}_1$. By Definition 4.4, $\mathcal{B}_2$ is not a continuation of $\mathcal{B}_1$ but a separate connected basin. □

Corollary 10.2 (ISM as Distinct Structural Basin)**.** *The Voyager 1 MAG post-crossing data (2013–2017) satisfies the conditions of Theorem 10.1:*

- *Class membership: 2013 and 2017 are 100% FULL; 2014–2016 average 97.3% FULL — comparable to or exceeding the heliosheath conformance.*
- $\overline{\kappa}_{\text{conn}}$*: post-crossing range 27,282–35,986 vs pre-crossing range 14,686–18,187 — distinct and non-overlapping.*
- $\overline{\text{tailDom}}$*: post-crossing range 0.932–0.947 vs pre-crossing 0.776–0.781 — distinct.*

Therefore the ISM is empirically realized as a distinct structural basin in $\mathcal{M}_{\text{adm}}$, not a continuation of heliosheath structure.

11 Two-Trajectory Structural Tomography

11.1 The Conjugate Role of Voyager 1 and Voyager 2

> **Key principle.** The Voyager pair forms a two-trajectory structural tomography of the heliosphere. Voyager 2 resolves the approach side of the heliopause through multi-variable plasma softening and localized FULL→GIANT onset, while Voyager 1 MAG resolves the crossing and post-crossing stabilization. The two trajectories are not redundant: they are complementary projections of the same boundary object in $\mathcal{M}_{\text{adm}}$.

This complementarity is summarized in Table 3.

Table 3: Two-Trajectory Complementarity: Voyager 2 and Voyager 1

Feature	**Voyager 2 (plasma, 2007–2018)**	**Voyager 1 (MAG, 2011–2017)**
Structural role	Pre-boundary approach	Crossing + post-boundary basin
Boundary epoch	2017–2018 approach	$t^* = 2012$ (estimator)
Dominant regime	FULL for V/T/w (96.0%)	FULL for $\|B\|$ (97.4%)
Boundary excursions	GIANT onset near heliopause	GIANT/TAIL/HARD clustered 2011–2012
κ_{conn} trend	Declining toward boundary	Minimum at t^*, ISM plateau
Tail dominance	Monotone decline ($0.556 \to 0.249$)	Sharp jump at crossing (+22%)
Observable	V, T, w, ρ (4 charts)	$\|B\|$ (1 chart)
Post-crossing	None	5 years of ISM (2013–2017)
Meaning	Transition layer detected before crossing	Transition layer crossed and resolved

11.2 The Two-Corridor Correlation Principle

Theorem 11.1 (Two-Corridor Correlation Principle)**.** *Independent spacecraft trajectories through the heliosphere need not share identical κ_{conn} profiles, but must correlate at the level of structural role:*

(i) ***Dominant-regime persistence:*** *each trajectory maintains a dominant realizability class throughout its traverse.*

(ii) ***Boundary-excursion clustering:*** *boundary-adjacent classifications (GIANT, TAIL) cluster near the heliopause approach epoch.*

(iii) ***Coordinate reorganization:*** *structural coordinates (κ_{conn}, tailDom) undergo directional reorganization near the physical boundary, either declining (pre-crossing approach) or rising (post-crossing stabilization).*

Both Voyager 2 and Voyager 1 satisfy all three conditions.

Remark 11.2. Theorem 11.1 formalizes the prediction from [Traj-V2] §9.2–9.3: different spacecraft do not probe different realizability spaces but trace distinct paths within the same $\mathcal{M}_{\text{adm}}$, correlating at the level of structural role rather than coordinate magnitude. This is precisely what the Voyager 1 MAG corpus confirms.

11.3 Two-Trajectory Boundary Reconstruction Theorem

Theorem 11.3 (Two-Trajectory Boundary Reconstruction)**.** *Let $\gamma_{\text{app}}(t)$ be a trajectory approaching a realizability boundary $\partial\mathcal{C}$ from one side, and let $\gamma_{\text{dep}}(t)$ be a trajectory departing from $\partial\mathcal{C}$ on the opposite side. Then:*

(i) $\kappa_{\text{conn,app}}(t)$ *is strictly decreasing as* $t \to t^*$*;*

(ii) $\kappa_{\text{conn,dep}}(t)$ *is strictly increasing as* $t \to t^*$*;*

(iii) The excursion sets $E_{\text{app}}(t)$ *and* $E_{\text{dep}}(t)$ *overlap in a neighbourhood of* t^**;*

(iv) The coordinate ranges on opposite sides are disjoint: $\mathcal{R}_{\text{app}} \cap \mathcal{R}_{\text{dep}} = \emptyset$.

Proof. From Theorem 6.1, a regular crossing produces a local minimum of κ_{conn} at t^*. Therefore, for $t < t^*$ sufficiently close, $\kappa_{\text{conn}}(t) > \kappa_{\text{conn}}(t^*)$ and the function must be decreasing as it approaches the minimum; for $t > t^*$, $\kappa_{\text{conn}}(t) > \kappa_{\text{conn}}(t^*)$ and the function must be increasing. This proves (i) and (ii).

From Theorem 8.1, boundary-adjacent classes concentrate in a neighbourhood of t^*. Since both γ_{app} and γ_{dep} cross the same boundary, their excursion sets must both be non-zero in that neighbourhood, proving (iii).

From Theorem 10.1, the pre- and post-crossing coordinate ranges are distinct and non-overlapping when the trajectory crosses into a different basin. Since γ_{app} terminates at $\partial\mathcal{C}$ and γ_{dep} begins there, and both are within a single regular crossing, the ranges on opposite sides are disjoint, proving (iv). □

Remark 11.4. Theorem 11.3 is the formal statement that the Voyager 1 and Voyager 2 corpora jointly reconstruct the same heliopause boundary object from opposite sides. This is the strongest form of the two-trajectory tomography claim.

12 Structural Diagnostics of the Heliospheric Boundary

We now elevate the empirical observations into a set of quantitative structural diagnostics. These are not merely descriptive; they constitute measurable criteria for identifying and characterizing a realizability phase transition.

12.1 Irreversibility of the Structural Transition

We test whether the structural trajectory returns to the pre-crossing coordinate band after the boundary. Define:

$$\mathsf{K}_{\text{HSH}} = [14,686,\ 18,187] \quad (2011\text{–}2012), \qquad \mathsf{K}_{\text{ISM}} = [27,282,\ 35,986] \quad (2013\text{–}2017).$$

These intervals are disjoint:

$$\mathsf{K}_{\text{HSH}} \cap \mathsf{K}_{\text{ISM}} = \emptyset.$$

The same holds for tail dominance:

$$\text{tailDom}_{\text{HSH}} \approx [0.776,\ 0.873], \qquad \text{tailDom}_{\text{ISM}} \approx [0.900,\ 0.947].$$

Therefore the post-crossing trajectory does not return to the pre-crossing structural band.

Proposition 12.1 (Structural Irreversibility)**.** *The heliopause crossing is structurally irreversible in observed coordinates: $\gamma(t > t^*)$ does not re-enter the pre-crossing coordinate range of κ_{conn} or* tailDom*. This strengthens Theorem 9.1 into a directional irreversibility result.*

12.2 Transition-Layer Thickness

Define the structural transition layer as the set of times where:

- κ_{conn} is depressed relative to the post-crossing baseline, and
- boundary-adjacent classes (GIANT, TAIL, HARD) are non-negligible.

From the data:

- 2011: elevated excursions + moderate κ_{conn},
- 2012: κ_{conn} minimum + excursions,
- 2013+: zero excursions + κ_{conn} plateau.

Thus the structural transition layer has finite thickness:

$$\mathcal{T}_{\text{HSH,ISM}} \approx [2011,\ 2012].$$

Proposition 12.2 (Finite Transition Layer)**.** *Realizability boundaries in $\mathcal{M}_{\text{adm}}$ are not measure-zero hypersurfaces in trajectory space; they induce finite-thickness transition layers. For the heliopause, $\mathcal{T}_{\text{HSH,ISM}}$ spans approximately one to two annual epochs.*

12.3 Basin Separation Index

We quantify basin separation using:

$$S_{\text{basin}} = \frac{\mu_{\text{ISM}} - \mu_{\text{HSH}}}{\sigma_{\text{ISM}} + \sigma_{\text{HSH}}}.$$

Using annual κ_{conn} values:

$$\mu_{\text{HSH}} \approx 16,437, \qquad \mu_{\text{ISM}} \approx 31,776, \qquad \Delta \approx 15,339.$$

Even without full σ evaluation, the separation satisfies $\Delta \gg \sigma_{\text{yearly}}$. The heliosheath and ISM therefore occupy well-separated structural basins in κ_{conn}-space.

12.4 Structural Jump Ratio

Define the post-crossing structural amplification:

$$R_\kappa = \frac{\kappa_{\text{post}}}{\kappa_{\text{min}}} = \frac{31,057}{14,686} \approx 2.11, \qquad R_\tau = \frac{\text{tailDom}_{\text{post}}}{\text{tailDom}_{\text{pre}}} \approx \frac{0.947}{0.776} \approx 1.22.$$

Proposition 12.3 (Structural Jump)**.** *The heliopause crossing produces a $\sim 2\times$ amplification in connectivity threshold and a $\sim 20\%$ increase in tail dominance.*

12.5 Boundary Sharpness Index

Define:

$$J_\kappa = \frac{\kappa_{\text{conn}}(t^+) - \kappa_{\text{conn}}(t^*)}{\kappa_{\text{conn}}(t^*)}.$$

For Voyager 1 MAG:

$$J_\kappa = \frac{31,057 - 14,686}{14,686} \approx 1.115.$$

Proposition 12.4 (Boundary Sharpness)**.** *The structural transition produces a* $+111.5\%$ *increase in* κ_{conn} *in a single annual step, indicating a sharp boundary rather than a gradual drift.*

12.6 Excursion-Collapse Index

Define the excursion density:

$$E(t) = \frac{N_{\text{GIANT}} + N_{\text{TAIL}} + N_{\text{HARD}}}{500}.$$

From the data:

$$E(2011) = 37/500 = 7.4\%, \qquad E(2012) = 26/500 = 5.2\%, \qquad E(2013) = 0/500 = 0\%.$$

Proposition 12.5 (Excursion Collapse)**.** *Boundary-adjacent regimes collapse immediately after the crossing:* $E(t) = 0$ *for all* $t > t^*$ *in the ISM basin.*

12.7 The Boundary Signature Triplet

The heliopause satisfies a minimal diagnostic signature:

(i) κ_{conn} minimum (2012);

(ii) Excursion clustering (GIANT/TAIL/HARD concentrated in 2011–2012);

(iii) Post-crossing plateau (κ_{conn} elevated, FULL dominant, zero excursions).

Proposition 12.6 (Boundary Signature Triplet)**.** *The triplet* (κ_{conn}*-minimum, excursion clustering, post-cro constitutes a measurable diagnostic criterion for identifying a realizability phase transition from structural data alone, without requiring external physical labels.*

12.8 Multi-Scale Robustness of the Structural Boundary Signature

To exclude segmentation artifacts as a source of the boundary signature, we evaluate the Voyager 1 MAG trajectory under three independent DLCP window configurations, holding all other pipeline parameters fixed:

$$(W, S) \in \{(512, 128), (1024, 256), (2048, 512)\}.$$

Data

For each configuration, 3,500 STRUC-PERC-I evaluations were produced (500 windows per annual epoch, 2011–2017) using identical B-component extraction, physical range filter

($0 < |B| < 10\,\text{nT}$), and sort+unique+finite adapter. Annualized structural coordinates were computed per epoch.

Table 4: Multi-Scale Structural Boundary Estimator — Annual Mean $\bar{\kappa}_{\text{conn}}$

Scale	2011	**2012**	2013	2014	2015	2016	2017	t^*
W512	6,303	**5,872**	15,432	13,895	12,439	13,404	14,752	**2012**
W1024	18,187	**14,686**	31,057	28,748	27,282	35,806	35,985	**2012**
W2048	49,504	**47,562**	69,380	81,292	73,038	84,475	82,383	**2012**

Bold: minimum per scale. All three scales identify $t^* = 2012$.

Table 5: Multi-Scale Robustness Summary

Feature	**W=512**	**W=1024**	**W=2048**
$t^* = 2012$	✓	✓	✓
Excursion peak 2011–2012	✓	✓	✓
κ_{conn} plateau post-2012	✓	✓	✓
Basin separation ($\mathcal{K}_{\text{pre}} \cap \mathcal{K}_{\text{post}} \approx \varnothing$)	✓	✓	✓
Jump ratio 2012→2013	2.63×	2.11×	1.46×
Post/crossing ratio	2.38	2.16	1.64

Scale-Invariant Boundary Estimator

Proposition 12.7 (Scale-Invariance of Boundary Estimator)**.** *The structural boundary estimator $t^* = \arg\min_t \bar{\kappa}_{\text{conn}}(t)$ is invariant under DLCP window scaling within the tested range $512 \leq W \leq 2048$:*

$$t^{*(512)} = t^{*(1024)} = t^{*(2048)} = 2012.$$

Proof. For each window size, the annual minimum of $\bar{\kappa}_{\text{conn}}(t)$ is uniquely attained at 2012 (Table 4). The ordering $\bar{\kappa}_{\text{conn}}(2012) < \bar{\kappa}_{\text{conn}}(t)$ for all $t \neq 2012$ is preserved across all three scales. Since t^* depends only on this ordering, not on the absolute magnitude of κ_{conn}, the estimator is scale-invariant within the tested range. □

Scale-Stable Boundary Signature Triplet

All three components of the triplet hold across all scales:

Proposition 12.8 (Triplet Robustness Under Window Scaling)**.** *For $(W, S) \in \{(512, 128), (1024, 256), (2048,$ the boundary signature triplet (Definition 4.5) is satisfied:*

(i) $\bar{\kappa}_{\text{conn}}$ has a unique minimum at $t^* = 2012$;

(ii) excursion density $E(t)$ is concentrated in 2011–2012 and collapses post-crossing;

(iii) post-crossing $\bar{\kappa}_{\text{conn}}$ occupies a higher, stable range.

Basin Separation Under Scale Variation

Define $\mathcal{K}_{\text{pre}} = \{\bar{\kappa_{\text{conn}}}(t) : t \le 2012\}$ and $\mathcal{K}_{\text{post}} = \{\bar{\kappa_{\text{conn}}}(t) : t > 2012\}$.

For W=512: $\mathcal{K}_{\text{pre}} = \{5{,}872, 6{,}303\}$; $\mathcal{K}_{\text{post}} = \{12{,}439, 13{,}404, 13{,}895, 14{,}752, 15{,}432\}$. No overlap.

For W=1024: $\mathcal{K}_{\text{pre}} = \{14{,}686, 18{,}187\}$; $\mathcal{K}_{\text{post}} = \{27{,}282, 28{,}748, 31{,}057, 35{,}806, 35{,}985\}$. No overlap.

For W=2048: $\mathcal{K}_{\text{pre}} = \{47{,}562, 49{,}504\}$; $\mathcal{K}_{\text{post}} = \{69{,}380, 73{,}038, 81{,}292, 82{,}383, 84{,}475\}$. No overlap.

Proposition 12.9 (Scale-Stable Basin Separation)**.** *Pre- and post-crossing κ_{conn} ranges are disjoint ($\mathcal{K}_{\text{pre}} \cap \mathcal{K}_{\text{post}} = \varnothing$) under all tested window scales.*

Interpretation

DLCP window size controls sampling locality, statistical smoothing, and ladder granularity, but does not alter the structural ordering of annual κ_{conn} means. The persistence of $t^* = 2012$ and the triplet across a fourfold variation in W implies that the detected transition reflects an intrinsic feature of the trajectory in $\mathcal{M}_{\text{adm}}$, not a segmentation artifact.

Remark 12.10. The absolute values of $\bar{\kappa_{\text{conn}}}$ scale with W (W=512: ∼6k–15k; W=1024: ∼15k–36k; W=2048: ∼48k–84k), as expected from window-size dependence of ladder cardinality. The *ordering* — and hence t^* and basin separation — is preserved.

Remark 12.11 (Scope of Robustness Claim)**.** The invariance demonstrated is empirical and restricted to $512 \le W \le 2048$. No claim is made regarding invariance under arbitrary segmentation, extrapolation outside this range, or coordinate-independent geometry.

13 Curvature and First Local Metric Reconstruction

Having established the boundary signature triplet and quantitative diagnostics, we now reconstruct the local geometry of the transition layer itself.

13.1 Boundary-Normal Coordinate

Using the structural boundary estimator $t^* = 2012$, the normalized boundary coordinate is $B_\kappa(t) = (\bar{\kappa_{\text{conn}}}(t) - \kappa_{\min})/(\kappa_{\text{post}} - \kappa_{\min})$. From Section 3:

$$B_\kappa(2011) = 0.205, \quad B_\kappa(2012) = 0, \quad B_\kappa(2013) = 0.958.$$

Thus B_κ behaves as a local normal coordinate to the boundary.

13.2 Discrete Curvature Estimator

Define the discrete structural curvature proxy:

$$\mathcal{K}_{\kappa\kappa}(t^*) = B_\kappa(t^* + 1) - 2B_\kappa(t^*) + B_\kappa(t^* - 1).$$

For Voyager 1:

$$\mathcal{K}_{\kappa\kappa}(2012) = 0.958 - 0 + 0.205 = 1.163.$$

Equivalently, in raw κ_{conn} units:

$$\Delta^2 \kappa_{\text{conn}}(2012) = 31,057 - 2(14,686) + 18,187 = 19,872.$$

Proposition 13.1 (Boundary Curvature)**.** *The heliopause boundary is not flat in structural coordinates: $\mathcal{K}_{\kappa\kappa}(t^*) = 1.163$. This is the first curvature proxy of a realizability boundary obtained from real physical trajectory data.*

Remark 13.2 (Curvature as Coordinate-Dependent Proxy)**.** The quantity $\mathcal{K}_{\kappa\kappa}(t^*)$ is a *discrete second-difference estimator in a fixed coordinate chart*, not an invariant geometric curvature of $\mathcal{M}_{\text{adm}}$. Specifically:

- It depends on the choice of normalized coordinate B_κ, which in turn depends on the choice of $\kappa_{\min}$ and κ_{post} as reference points.
- It is computed along a single trajectory in a single observable channel ($|B|$); it is not a curvature of the manifold $\mathcal{M}_{\text{adm}}$ itself.
- Computing an invariant Riemannian curvature scalar for $\mathcal{M}_{\text{adm}}$ would require a full tensor metric, multiple independent trajectories, and coordinate-change consistency — none of which are currently available.

$\mathcal{K}_{\kappa\kappa}(t^*)$ should therefore be interpreted as a *local trajectory curvature proxy* that characterizes the shape of the crossing in one coordinate direction. It is the first such proxy from real data, but it is not an intrinsic geometric invariant.

13.3 Slope Asymmetry

The incoming and outgoing normalized slopes are:

$$s_- = B_\kappa(2012) - B_\kappa(2011) = -0.205, \qquad s_+ = B_\kappa(2013) - B_\kappa(2012) = 0.958.$$

Define the asymmetry ratio:

$$A_\kappa = \frac{|s_+|}{|s_-|} = \frac{0.958}{0.205} \approx 4.67.$$

Proposition 13.3 (Slope Asymmetry)**.** *The trajectory does not leave the boundary at the same rate at which it approached it. The boundary is directionally curved/asymmetric in structural coordinates.*

13.4 Local Metric Tensor Reconstruction

Define the local structural coordinate vector:

$$z(t) = \big(B_\kappa(t),\ B_\tau(t),\ E(t),\ 1 - \text{GR}(t)\big),$$

where $B_\tau(t) = (\text{tailDom}(t) - \text{tailDom}_{\min})/(\text{tailDom}_{\text{post}} - \text{tailDom}_{\min})$. For the first reconstruction, take the normalized diagonal gauge:

$$ds^2 = dB_\kappa^2 + dB_\tau^2 + dE^2 + d(1 - \text{GR})^2.$$

Across the 2012–2013 crossing:

$$\Delta B_\kappa = 0.958, \quad \Delta B_\tau \approx 1.04, \quad \Delta E = 0.052, \quad \Delta(1 - \mathrm{GR}) \approx 0.0003 \text{ (negligible)}.$$

Thus:

$$ds_{\mathrm{cross}} = \sqrt{0.958^2 + 1.04^2 + 0.052^2} \approx 1.42.$$

Proposition 13.4 (First Local Metric Reconstruction)**.** *The heliopause crossing has finite structural length in local $\mathcal{M}_{\mathrm{adm}}$ coordinates: $ds_{\mathrm{cross}} \approx 1.42$. This is a first empirical distance estimate between the boundary point and the ISM basin.*

Lemma 13.5 (Empirical Local Metric Tensor)**.** *Given a cloud of structural trajectory points $\{z_i\}$ in a local chart, the empirical metric tensor can be estimated by $g \approx \Sigma^{-1}$, where $\Sigma = \mathrm{Cov}(z_i)$ is the covariance matrix of structural coordinates within a basin or transition layer. Large variance directions correspond to structurally soft directions; small variance directions correspond to structurally stiff directions; off-diagonal terms indicate coupling between coordinates.*

Remark 13.6. The tensor $g \approx \Sigma^{-1}$ is an empirical local metric proxy defined in a fixed coordinate chart. It is **not invariant under reparameterization** of $\mathcal{M}_{\mathrm{adm}}$ (see Remark 13.8). Results derived from it should be read as directional stiffness indicators in the chosen chart, not as intrinsic Riemannian geometry.

Remark 13.7. Current Voyager data supports local determination of $g_{\kappa\kappa}$, $g_{\tau\tau}$, and g_{EE} for the heliopause transition. Full determination of the global tensor g_{ij} on $\mathcal{M}_{\mathrm{adm}}$ requires more independent crossings, multiple observables on both sides, and cross-domain trajectories — a limitation consistent with the manuscript's stated scope.

Remark 13.8 (Metric Tensor as Empirical Local Proxy — Non-Invariance)**.** The estimator $g \approx \Sigma^{-1}$ is an *empirical local metric proxy* defined in a fixed coordinate chart. It is **not coordinate-invariant**: under a reparameterization $\tilde{z} = f(z)$ of the structural observables, the covariance matrix transforms as $\widetilde{\Sigma} = J\Sigma J^\top$ (where $J = \partial f/\partial z$), so $\tilde{g} = \widetilde{\Sigma}^{-1} \neq g$ in general. Therefore $g \approx \Sigma^{-1}$ should be read as a directional stiffness/softness indicator in the chosen chart, not as a Riemannian metric tensor on $\mathcal{M}_{\mathrm{adm}}$. A coordinate-invariant metric would require a diffeomorphism-covariant construction, which is not yet established for $\mathcal{M}_{\mathrm{adm}}$.

Remark 13.9 (Coordinate Dependence of All Local Geometry Quantities)**.** All curvature and metric quantities reported in this section ($\mathcal{K}_{\kappa\kappa}(t^*) = 1.163$, $A_\kappa \approx 4.67$, $ds_{\mathrm{cross}} \approx 1.42$) are chart-dependent observables, not invariant geometric quantities of $\mathcal{M}_{\mathrm{adm}}$ itself. They depend on the choice of normalized coordinates and the diagonal gauge. A coordinate-invariant curvature scalar would require a full Riemannian metric on $\mathcal{M}_{\mathrm{adm}}$, which is not yet available. These quantities should therefore be interpreted as empirical proxies for geometric structure, not as absolute geometric invariants.

14 First Basin Atlas of $\mathcal{M}_{\mathrm{adm}}$

The Global Phase Structure Conjecture states that $\mathcal{M}_{\mathrm{adm}} = \bigcup_\alpha \mathcal{B}_\alpha \cup \bigcup_{\alpha,\beta} T_{\alpha\beta}$. The Voyager pair supplies one pre-boundary trajectory, one crossing, and one post-boundary basin. Together with prior static UNNS domains, we can now construct the first empirical basin atlas.

Table 6: First Basin Atlas of $\mathcal{M}_{\mathrm{adm}}$

Basin	Source	Class	Structural range
$\mathcal{B}_{\mathrm{bio,FULL}}$	QT45 fitness landscapes	FULL	$\kappa_{\mathrm{conn}} \approx 0.42$–$2.00$ (soft biological basin)
$\mathcal{B}_{\mathrm{plasma,FULL}}$	Voyager 2 V/T/w	FULL	$\kappa_{\mathrm{conn}} \approx 50$–$2200$ (heliosheath plasma corridor)
$\mathcal{B}_{\mathrm{HSH,MAG}}$	Voyager 1 2011–2012	FULL + excursions	$\kappa_{\mathrm{conn}} \approx$ 14k–18k (heliosheath magnetic boundary basin)
$\mathcal{T}_{\mathrm{HSH,ISM}}$	Voyager 1 2011–2012	GIANT/TAIL/HARD enriched	κ_{conn} minimum, excursion cluster (heliopause transition layer)
$\mathcal{B}_{\mathrm{ISM,MAG}}$	Voyager 1 2013–2017	FULL	$\kappa_{\mathrm{conn}} \approx$ 27k–36k (post-crossing ISM basin)

Corollary 14.1 (Basin Stratification)**.** *The admissibility manifold $\mathcal{M}_{\mathrm{adm}}$ is not merely partitioned into realizability classes, but stratified into metric basins within classes. Class labels define coarse topology; basin coordinates define local geometry. The same class (FULL) contains at least four distinct basins with non-overlapping κ_{conn} ranges: soft biological, heliosheath plasma, heliosheath magnetic, and ISM magnetic.*

15 Negative Control Analysis

15.1 Objective

The boundary signature triplet (Definition 4.5) is claimed to be a diagnostic indicator of realizability boundary crossings. To establish diagnostic validity, it is necessary to demonstrate that the triplet does NOT appear in the absence of a boundary.

15.2 Definition: Boundary Signature Absence

Definition 15.1 (Absence of Boundary Signature)**.** Let $\gamma(t) \subset \mathcal{M}_{\mathrm{adm}}$ be a structural trajectory. We say that γ exhibits *no boundary signature* if:

1. $\kappa_{\mathrm{conn}}(t)$ has no isolated local extremum;
2. $E(t) \equiv 0$ (or remains uniformly negligible, $E(t) < \epsilon$ for all t);
3. structural coordinates remain within a single bounded range without basin transition.

15.3 Synthetic Monotonic Control

Definition 15.2 (Monotonic Control Construction)**.** Let $x(t) = at + b$ with $a \neq 0$. Define ladders $L(t)$ from monotonic sequences of $x(t)$ without structural perturbation. Under sorting, such sequences produce a single dominant connected component, no secondary clusters, and no tail amplification.

Proposition 15.3 (Monotonic Control Yields No Boundary Signature)**.** *For monotonic input signals $x(t) = at + b$, the induced structural trajectory satisfies:*

$$E(t) = 0 \quad \forall t,$$

and $\kappa_{\text{conn}}(t)$ is monotonic or constant. Hence it satisfies Definition 15.1.

Proof. Monotonic sequences produce, under sorting, a ladder with a single dominant gap structure. No fragmentation occurs; no secondary clusters form; no tail structures develop. Therefore:

- The FULL class persists throughout;
- Excursion classes (GIANT, TAIL, HARD) are never triggered;
- $E(t) = 0$ identically;
- $\kappa_{\text{conn}}(t)$ varies monotonically with the range of $x(t)$ or remains constant.

Thus all three conditions of Definition 15.1 are satisfied. □

15.4 Stable ISM Segment Control

Proposition 15.4 (ISM Segment Yields No Boundary Signature)**.** *Consider the post-crossing interval $t \in [2013, 2017]$ from Voyager 1 MAG. This trajectory segment satisfies:*

$$E(t) = 0 \quad \forall t \in [2013, 2017],$$

and $\kappa_{\text{conn}}(t) \in [27, 282, 35, 986]$ with no return to pre-crossing values. Hence it satisfies Definition 15.1.

Proof. Empirical observation from Table 2:

- 2013: 500 FULL, 0 GIANT, 0 TAIL, 0 HARD $\Rightarrow E = 0$;
- 2014: 495 FULL, 4 GIANT, 1 TAIL, 0 HARD $\Rightarrow E \approx 0.01$ (negligible);
- 2015: 489 FULL, 5 GIANT, 6 TAIL, 0 HARD $\Rightarrow E \approx 0.022$ (negligible);
- 2016: 488 FULL, 10 GIANT, 2 TAIL, 0 HARD $\Rightarrow E \approx 0.024$ (negligible);
- 2017: 500 FULL, 0 GIANT, 0 TAIL, 0 HARD $\Rightarrow E = 0$.

$E(t)$ remains uniformly negligible (< 0.025) throughout the ISM segment. $\kappa_{\text{conn}}(t)$ fluctuates but remains strictly within the ISM basin range with no isolated extremum. Therefore the segment exhibits no boundary signature. □

15.5 Negative Control Theorem

Theorem 15.5 (Non-Universality of the Boundary Signature)**.** *The boundary signature triplet does not occur for all structural trajectories. There exist trajectories (synthetic monotonic, stable ISM) for which the triplet is absent.*

Proof. From Proposition 15.3, synthetic monotonic trajectories lack κ_{conn} extremum, have $E(t) = 0$, and show no basin transition. From Proposition 15.4, the ISM segment lacks all three signature components. Hence the triplet is not a universal feature of all trajectories. □

15.6 Diagnostic Implication

Corollary 15.6 (Signature as Necessary Indicator)**.** *The boundary signature triplet is a necessary indicator of a realizability boundary crossing, not a universal feature of all trajectories. If a trajectory exhibits the triplet, it provides evidence of a boundary crossing; if it does not, no conclusion about boundary presence can be drawn (the trajectory may be stable or may cross without producing the triplet — see falsification criteria in Section 16).*

16 Falsifiability of the Structural Boundary Framework

16.1 Objective

A structural theory must admit explicit falsification conditions. This section defines conditions under which the proposed framework fails.

16.2 Falsifiability Criteria

Criterion 16.1 (Failure of Critical Point)**.** A trajectory crosses a physical boundary but $\kappa_{\text{conn}}(t)$ has no local extremum (and no discontinuity) at the crossing epoch.

Criterion 16.2 (Failure of Excursion Localization)**.** A trajectory exhibits a boundary crossing but $E(t)$ does NOT localize near the boundary region (i.e., excursions are uniformly distributed or absent).

Criterion 16.3 (Failure of Basin Separation)**.** Post-crossing structural coordinates satisfy:

$$\mathsf{K}_{\text{pre}} \cap \mathsf{K}_{\text{post}} \neq \emptyset,$$

where K_{pre} and K_{post} are the coordinate ranges before and after the crossing.

Criterion 16.4 (Failure of Structural Transition)**.** No measurable discontinuity or reorganization occurs in structural coordinates across the supposed physical boundary (i.e., κ_{conn}, tailDom, and GR are statistically indistinguishable on both sides).

16.3 Falsifiability Theorem

Theorem 16.5 (Structural Falsifiability)**.** *If any of Criteria 16.1–16.4 are satisfied in a system with a confirmed physical boundary, then the structural boundary framework is invalid for that system.*

Proof. Each criterion negates one component of the boundary signature triplet (Definition 4.5):

- Criterion 16.1 negates the κ_{conn} extremum condition;
- Criterion 16.2 negates the transition layer condition;

- Criterion 16.3 negates the basin separation condition;
- Criterion 16.4 negates the requirement of structural reorganization across the boundary.

If any component fails for a confirmed boundary crossing, the framework's diagnostic claim is falsified for that system. □

16.4 Domain-Level Falsification

The framework is falsified at the domain level if there exists a domain D such that:

$$\text{boundary present in } D \ \wedge \ \text{boundary signature triplet absent in } D.$$

16.5 Practical Tests

To attempt falsification of the theory, one may:

1. Analyze systems with known phase transitions (thermodynamic, quantum, structural, biological);
2. Compute DLCP-based structural trajectories for these systems;
3. Verify presence or absence of the boundary signature triplet (Definition 4.5) at the known transition epoch.

16.6 Interpretation

The framework is:

- **Not descriptive** — it does not apply universally to all trajectories (Section 15 demonstrates non-universality);
- **Not tautological** — it can fail (Criteria 16.1 to 16.4 provide explicit failure modes);
- **Not unfalsifiable** — explicit counterexamples would refute it.

16.7 Final Statement on Falsifiability

> The boundary signature framework is a testable structural hypothesis:
>
> $$\text{physical boundary crossing} \ \Rightarrow \ \text{boundary signature triplet}$$
>
> subject to the assumptions of regular crossing, DLCP windowing validity, and coordinate continuity. The converse does not necessarily hold: the triplet may appear in other contexts (e.g., near-misses), and its absence may occur for reasons other than boundary absence. The framework is falsified if a confirmed boundary crossing fails to produce the triplet under the stated assumptions.

17 Geometry and Universality of Realizability Boundaries in $\mathcal{M}_{\text{adm}}$

Section 16. This section establishes that realizability boundaries in the admissibility manifold $\mathcal{M}_{\text{adm}}$ possess: (i) geometric structure (curvature, metric separation); (ii) dynamic realizability (Voyager trajectories); and (iii) cross-domain replication (atomic and cosmological datasets). The central result is that realizability boundaries are geometric objects whose structural signatures are empirically universal across independent domains.

17.1 Preliminaries

Let $\gamma : I \to \mathcal{M}_{\text{adm}}$ be a structural trajectory. Let $z(t) = (\kappa_{\text{conn}}(t), \text{tailDom}(t), E(t), \text{GR}(t))$ be continuous in t except at possible chart transitions. We assume:

- (A1) Local chart regularity near boundaries [LocalGeom];
- (A2) Margin-distance equivalence [LocalGeom];
- (A3) Continuity of observables under DLCP windowing [Traj-V2].

17.2 Boundary Signature

Definition 17.1 (Dynamic Boundary Signature)**.** Let $\gamma(t) \subset \mathcal{M}_{\text{adm}}$ be a structural trajectory. A point $t^* \in I$ satisfies the *dynamic boundary signature* if:

1. $\kappa_{\text{conn}}(t^*) = \min_{t \in I} \kappa_{\text{conn}}(t)$ (unique local minimum);
2. $\exists \delta > 0$ such that $E(t) > 0$ for all $t \in (t^* - \delta, t^* + \delta)$;
3. $\exists \epsilon > 0$ such that $\kappa_{\text{conn}}(t) \geq \kappa_{\text{conn}}(t^*) + \epsilon$ for all $t > t^* + \delta$.

Definition 17.2 (Static Boundary Signature)**.** Let $\gamma(\lambda) \subset \mathcal{M}_{\text{adm}}$ be a parameterized structural path. A point λ^* satisfies the *static boundary signature* if:

1. $\kappa_{\text{conn}}(\lambda)$ admits a local extremum or discontinuity at λ^*;
2. $E(\lambda) > 0$ in a neighbourhood of λ^*;
3. Structural coordinates undergo a discontinuous or non-smooth change across λ^*.

Theorem 17.3 (Existence of Critical Point under Crossing)**.** *If a trajectory $\gamma(t)$ crosses a realizability boundary ∂C transversally, then there exists $t^* \in I$ such that $\frac{d}{dt}\kappa_{\text{conn}}(t^*) = 0$.*

Proof. From [LocalGeom], $m(L) \sim d_{\partial C}(L)$. Under crossing, $d_{\partial C}(t^*) = 0$ and $d(t) > 0$ for $t \neq t^*$. Thus $d(t)$ attains a local minimum at t^*. Since $\kappa_{\text{conn}}(t) = f(m(t))$ with $f' > 0$ (margin monotonic in the active branch), $\kappa_{\text{conn}}(t)$ also attains a local minimum at t^*, so $\frac{d}{dt}\kappa_{\text{conn}}(t^*) = 0$. □

17.3 Voyager Realization

Theorem 17.4 (Voyager Boundary Signature). *The Voyager 1 MAG trajectory $\gamma_{V1}(t)$ satisfies the dynamic boundary signature at $t^* = 2012$.*

Proof. Empirical verification from Section 2.2:

1. $\bar{\kappa_{\text{conn}}}(2012) = 14,686$ is the unique annual minimum;
2. $E(t) > 0$ for $t \in [2011, 2012]$ (excursion density 5.2–7.4%);
3. For $t > 2012$, $\bar{\kappa_{\text{conn}}}(t) \geq 27,282$, with $\epsilon = 12,596$.

All three conditions of Definition 17.1 are satisfied. □

17.4 Curvature of Realizability Boundaries

Definition 17.5 (Normalized Boundary Coordinate). Define:

$$B_\kappa(t) = \frac{\kappa_{\text{conn}}(t) - \kappa_{\min}}{\kappa_{\text{post}} - \kappa_{\min}},$$

where $\kappa_{\min} = \min_t \kappa_{\text{conn}}(t)$ and κ_{post} is the post-crossing baseline.

Definition 17.6 (Discrete Curvature Proxy).

$$\mathcal{K}_{\kappa\kappa}(t^*) = B_\kappa(t^* + 1) - 2B_\kappa(t^*) + B_\kappa(t^* - 1).$$

Proposition 17.7 (Curvature Estimate). *For Voyager 1 MAG: $\mathcal{K}_{\kappa\kappa}(2012) = 1.163$.*

Proof. From Section 13: $B_\kappa(2011) = 0.205$, $B_\kappa(2012) = 0$, $B_\kappa(2013) = 0.958$. Direct substitution yields:

$$\mathcal{K}_{\kappa\kappa}(2012) = 0.958 - 2(0) + 0.205 = 1.163.$$

□

Proposition 17.8 (Slope Asymmetry). *Define $A_\kappa = |B_\kappa(t^* + 1)|/|B_\kappa(t^* - 1)|$. Then $A_\kappa \approx 4.7$.*

Theorem 17.9 (Boundary Asymmetry). *Realizability boundaries in $\mathcal{M}_{\text{adm}}$ are not symmetric under trajectory reversal in structural coordinates.*

Proof. If $A_\kappa \neq 1$, then $s_- \neq -s_+$, where $s_- = B_\kappa(t^*) - B_\kappa(t^* - 1)$ and $s_+ = B_\kappa(t^* + 1) - B_\kappa(t^*)$. Thus the trajectory is not invariant under $t \mapsto 2t^* - t$. □

17.5 Metric Structure

Definition 17.10 (Local Structural Coordinates).

$$z = (B_\kappa, B_\tau, E, 1 - \text{GR}),$$

where B_τ is the normalized tail dominance coordinate.

Definition 17.11 (Empirical Metric Proxy). Let Σ be the covariance matrix of $\{z_i\}$ in a neighbourhood. Define $g = \Sigma^{-1}$. This defines a positive-definite bilinear form.

Proposition 17.12 (Finite Structural Distance). *Across the Voyager 1 crossing, $ds_{\text{cross}} \approx 1.42$.*

Proof. In the diagonal gauge, $ds^2 = dB_\kappa^2 + dB_\tau^2 + dE^2 + d(1 - \text{GR})^2$. From Section 13: $\Delta B_\kappa = 0.958$, $\Delta B_\tau \approx 1.04$, $\Delta E = 0.052$, $\Delta(1 - \text{GR}) \approx 0.0003$. Hence:

$$ds_{\text{cross}} = \sqrt{0.958^2 + 1.04^2 + 0.052^2} \approx 1.42.$$

□

Theorem 17.13 (Finite Transition Length). *Realizability boundary crossings correspond to finite-length segments in $\mathcal{M}_{\text{adm}}$ under the empirical metric proxy.*

17.6 Atomic Domain

Proposition 17.14 (Structural Sequence under Deformation). *Atomic spectra under Zeeman deformation satisfy:*

$$\textit{HARD} \rightarrow \textit{FULL} \rightarrow \textit{GIANT/TAIL}.$$

Theorem 17.15 (Partial Boundary Signature in Atomic Domain). *Atomic structural paths exhibit excursion emergence and regime transition, satisfying a partial static boundary signature.*

Proof. As deformation parameter λ increases:

- Connectivity weakens, GR decreases;
- Tail components increase, tailDom $\rightarrow 1$;
- Excursions (GIANT/TAIL classes) emerge.

Thus conditions (ii) and (iii) of Definition 17.2 are satisfied, though (i) is not resolved due to lack of continuous trajectory sampling. □

17.7 Cosmological Domain

Proposition 17.16 (κ_{conn}-Cliff Detection). *Cosmological ladders exhibit a threshold $\kappa_c \approx 0.307$ such that:*

$$\kappa < \kappa_c \Rightarrow \textit{stable regime}, \qquad \kappa > \kappa_c \Rightarrow \textit{collapse regime}.$$

Theorem 17.17 (Static Boundary Signature in Cosmological Domain). *Cosmological κ_{conn}-sweeps satisfy the static boundary signature.*

Proof. From Section 2.4:

- κ_{conn} cliff at $\kappa_c \approx 0.307$ (discontinuity in ν);
- Instability region at $\kappa \approx 0.273$ (excursion analogue);

- Post-collapse regime with distinct structural coordinates.

All three conditions of Definition 17.2 are satisfied. □

17.8 Universality

Theorem 17.18 (Boundary Signature Universality (Empirical Principle))**.** *Across the Voyager (dynamic), atomic (parameter sweep), and cosmological (κ_{conn}-sweep) datasets, realizability boundaries satisfy:*

1. *Concentration of non-FULL classes (excursions) localized near the boundary;*
2. *Rapid structural reorganization across the transition;*
3. *Separation of pre- and post-transition regimes into distinct structural basins with non-overlapping coordinate ranges.*

These features are observed across physical scale, observable type, and domain-specific dynamics in the corpora examined.

Remark 17.19 (Status of the Universality Claim)**.** Theorem 17.18 is an *empirical principle*, not a deductive theorem. Its "proof" is by enumeration of corpora sharing a common protocol (DLCP, STRUC-PERC-I, PRP thresholds): all evaluated datasets exhibit the three-item pattern. This establishes the pattern as *robust across the tested corpora*, not as a necessary logical consequence of the UNNS axioms. In particular:

- the claim does not exclude future domains where the pattern fails;
- the cross-domain invariance rests on the shared evaluation protocol, not on a domain-independent geometric proof;
- a deductive proof of the pattern from the PRP axioms alone remains open.

The result is therefore best understood as a *cross-domain empirical regularity* supporting the Global Phase Structure Conjecture, not as a proved theorem of the framework.

17.9 Extended Basin Atlas

Table 7: Extended Basin Atlas with Cross-Domain Ordering

Basin	**Source**	**Class**	κ_{conn} range	**Structural role**
$\mathcal{B}_{\text{bio,FULL}}$	QT45	FULL	0.42–2.00	Soft biological interior
$\mathcal{B}_{\text{plasma,FULL}}$	V2 V/T/w	FULL	50–2200	Heliosheath plasma corridor
$\mathcal{B}_{\text{HSH,MAG}}$	V1 2011–2012	FULL + excursions	14k–18k	Pre-crossing magnetic basin
$\mathcal{T}_{\text{HSH,ISM}}$	V1 2011–2012	GIANT/TAIL/HARD	κ_{conn} minimum	Heliopause transition layer
$\mathcal{B}_{\text{ISM,MAG}}$	V1 2013–2017	FULL	27k–36k	Post-crossing ISM basin
$\mathcal{B}_{\text{atomic,FULL}}$	Li/Na QM-I	FULL	$\sim 10^5$	Quantum interior basin
$\mathcal{B}_{\text{atomic,TAIL}}$	Au Zeeman	TAIL	$\sim 10^5$	Deformation-excursion basin
$\mathcal{B}_{\text{cosmo,pre}}$	DESI plateau	FULL	$\kappa_{\text{conn}} \approx 0.19$	Pre-cliff basin
$\mathcal{T}_{\text{cosmo}}$	DESI cliff	GIANT analogue	$\kappa_{\text{conn}} \approx 0.27$	Void-filament transition
$\mathcal{B}_{\text{cosmo,post}}$	DESI collapsed	Fragmented	$\kappa_{\text{conn}} > 0.30$	Post-collapse regime

17.10 Synthesis Theorem

Theorem 17.20 (Geometry-Universality Correspondence). *Realizability boundaries in $\mathcal{M}_{\text{adm}}$ satisfy:*

1. ***Geometric structure:***
   - *finite curvature* ($\mathcal{K}_{\kappa\kappa}(t^*) = 1.163$);
   - *finite metric thickness* ($ds_{\text{cross}} \approx 1.42$);
   - *directional asymmetry* ($A_\kappa \approx 4.7$).
2. ***Dynamic realizability:***
   - *trajectories exhibit the boundary signature triplet;*
   - *excursions concentrate near the crossing;*
   - *post-crossing basins have non-overlapping coordinate ranges.*
3. ***Cross-domain replication:***
   - *boundary signatures replicate across heliospheric, atomic, and cosmological domains;*
   - *the replication is structural, not physical: scales and observables differ, but the $\mathcal{M}_{\text{adm}}$ boundary pattern is identical.*

Proof. (1) follows from Propositions 17.7, 17.12, and 17.8. (2) follows from Theorem 17.4 and Proposition 12.6. (3) follows from Theorem 17.18. Thus all three properties hold simultaneously. □

17.11 Final Statement on Universality

The combined Voyager, atomic, and cosmological corpora establish that:

> Realizability boundaries in $\mathcal{M}_{\text{adm}}$ are curved, finite-thickness geometric objects whose structural signatures — κ_{conn} extremum, excursion clustering, basin separation — are empirically universal across independent physical domains. The boundary behaviour is not domain-specific; it is an empirically stable structural property across the tested domains, detectable through structural diagnostics alone.

17.12 Limitations (Cross-Domain)

This section does *not* establish:

- full metric equivalence across domains (the coordinate ranges are not directly comparable);
- identical curvature values across domains (only structural homology);
- a complete atlas of $\mathcal{M}_{\text{adm}}$ (only the first empirical basins);

- causality between domains (they are independent datasets);
- coordinate-invariant geometry (all curvature and metric quantities are chart-dependent observables).

What it establishes is *structural homology* of boundary signatures across independent physical domains — strong evidence that realizability boundaries are universal geometric objects in $\mathcal{M}_{\text{adm}}$.

17.13 Scale-Invariance of Structural Boundary Detection

The robustness analysis of Section 12.8 allows elevation of the empirical observation to a structural constraint on admissible trajectories.

Theorem 17.21 (Scale-Invariance of Boundary Estimator). *Let $\gamma(t) \subset \mathcal{M}_{\text{adm}}$ be a structural trajectory obtained via DLCP ladder construction from a fixed physical dataset. Let $t^{*(W)}$ denote the boundary estimator obtained using window size W. Then for all tested scales $W \in \{512, 1024, 2048\}$,*

$$t^{*(W)} \;=\; t^*,$$

i.e., the boundary estimator is invariant under DLCP window scaling within the tested range $512 \leq W \leq 2048$.

Proof. Section 12.8 establishes that for all tested window sizes,

$$\arg\min_t \bar{\kappa_{\text{conn}}}^{(W)}(t) \;=\; 2012.$$

Since the ordering of annualized connectivity values is preserved across all tested scales (Table 4), and t^* depends only on this ordering, the estimator is invariant. □

Corollary 17.22 (Segmentation-Independence). *The structural boundary signature is not an artifact of temporal segmentation within the tested scale range. The boundary localization reflects an intrinsic property of the trajectory in $\mathcal{M}_{\text{adm}}$ rather than a property of the DLCP construction.*

Remark 17.23. Scale-invariance is empirical and restricted to $512 \leq W \leq 2048$. No claim is made regarding invariance under arbitrary segmentation or coordinate transformations.

17.14 Structural Boundary Operator

The preceding results motivate a formal operator that maps a structural trajectory to its detected boundary epoch.

Definition 17.24 (Structural Boundary Operator). Let $\gamma(t) \subset \mathcal{M}_{\text{adm}}$ be a structural trajectory obtained from DLCP ladders and STRUC-PERC-I evaluation. Define the structural observable vector

$$Z(t) \;=\; \big(\bar{\kappa_{\text{conn}}}(t),\; E(t),\; \text{tailDom}(t),\; \text{GR}(t)\big).$$

The *Structural Boundary Operator* is the map

$$\mathcal{B}_{\text{op}} \;:\; \gamma \;\longmapsto\; t^*,$$

defined by

$$\mathcal{B}_{\mathrm{op}}[\gamma] \;=\; \arg\min_{t} \bar{\kappa}_{\mathrm{conn}}(t),$$

provided the following *admissibility conditions* hold:

(i) the minimum is unique;

(ii) excursion density $E(t)$ is localized in a neighbourhood of t^*;

(iii) post-boundary coordinates occupy a range disjoint from pre-boundary coordinates.

If these conditions fail, $\mathcal{B}_{\mathrm{op}}[\gamma]$ is undefined.

Theorem 17.25 (Boundary Operator Correctness)**.** *Let $\gamma(t)$ be a regular structural trajectory crossing a realizability boundary $\partial\mathcal{C}$ transversally at $t = t^*$. Assume margin-boundary equivalence (Theorem 5.1), threshold continuity (Theorem 8.1), and basin separation (Theorem 10.1). Then*

$$\mathcal{B}_{\mathrm{op}}[\gamma] \;=\; t^*.$$

Proof. By Theorem 5.1, the connectivity margin is locally order-equivalent to boundary distance. At a regular crossing, boundary distance attains a strict local minimum at t^*; therefore $\bar{\kappa}_{\mathrm{conn}}(t)$ attains its unique minimum at t^*, satisfying condition (i). By Theorem 8.1, boundary-adjacent excursions concentrate near t^*, satisfying condition (ii). By Theorem 10.1, post-crossing coordinates occupy a distinct basin with non-overlapping κ_{conn} range, satisfying condition (iii). All admissibility conditions hold, so the operator returns t^*. □

Corollary 17.26 (Scale-Stable Boundary Operator)**.** *Let $\mathcal{B}_{\mathrm{op}}^{(W)}$ denote the boundary operator applied to DLCP ladders constructed with window size W. For the Voyager 1 MAG corpus (Proposition 12.7),*

$$\mathcal{B}_{\mathrm{op}}^{(512)}[\gamma] \;=\; \mathcal{B}_{\mathrm{op}}^{(1024)}[\gamma] \;=\; \mathcal{B}_{\mathrm{op}}^{(2048)}[\gamma] \;=\; 2012.$$

The detected boundary epoch is stable under tested DLCP window scaling.

Remark 17.27 (Pre-Specification)**.** The operator $\mathcal{B}_{\mathrm{op}}$ is not fitted to data. It is fully specified prior to evaluation — by the admissibility conditions of Definition 17.24 — and applied identically across all datasets. The result $t^* = 2012$ follows from the data, not from tuning.

17.15 Failure Modes of the Structural Boundary Operator

Proposition 17.28 (Failure Conditions of $\mathcal{B}_{\mathrm{op}}$)**.** *The operator $\mathcal{B}_{\mathrm{op}}[\gamma]$ is undefined (returns no output) if any of the following occur:*

(i) $\bar{\kappa}_{\mathrm{conn}}(t)$ has no unique minimum — the trajectory does not approach a realizability boundary, or multiple boundary approaches occur simultaneously;

(ii) excursion density $E(t)$ is not localized near the candidate minimum — the triplet is absent or distributed uniformly, indicating no transition layer;

(iii) *pre- and post-candidate coordinate distributions overlap significantly — no basin separation exists and the candidate t^* does not correspond to a structural boundary.*

In such cases the trajectory either does not cross a realizability boundary within the observation window, or the boundary crossing cannot be resolved at the given DLCP scale.

Remark 17.29 (Multi-Crossing Case)**.** If $\bar{\kappa_{\text{conn}}}(t)$ admits multiple local minima, $\mathcal{B}_{\text{op}}$ selects the global minimum by default. Multiple boundary crossings correspond to multiple admissible minima and require segmentation of γ into sub-intervals. Each sub-interval is then processed independently to resolve the distinct crossings. This is the natural extension to multi-phase trajectories.

17.16 Stability Hypothesis

Proposition 17.30 (Stability under Trajectory Perturbation — Hypothesis)**.** *Let $\gamma_\epsilon(t)$ be a perturbation of a structural trajectory $\gamma(t)$ such that $\|\gamma - \gamma_\epsilon\|_{\mathcal{M}_{\text{adm}}} < \epsilon$ in the product metric on annualized structural coordinates. Then there exists $\delta(\epsilon) \geq 0$ with $\delta \to 0$ as $\epsilon \to 0$ such that*

$$\left|\mathcal{B}_{\text{op}}[\gamma] - \mathcal{B}_{\text{op}}[\gamma_\epsilon]\right| \;\leq\; \delta(\epsilon),$$

provided both operators are defined.

Remark 17.31. This is stated as a hypothesis, not a proved theorem. It asserts that $\mathcal{B}_{\text{op}}$ is not brittle pattern detection: small perturbations of the trajectory produce small or zero changes in the detected boundary epoch. The multi-scale robustness result (Propositions 12.7 and 12.8) provides empirical support: the fourfold variation in W constitutes a structured perturbation of the trajectory that leaves t^* unchanged. A deductive proof of the hypothesis from the PRP axioms remains open.

17.17 Algorithm

For practical application, the Structural Boundary Operator reduces to the following procedure:

Algorithm: Structural Boundary Detection via $\mathcal{B}_{\text{op}}$

Input: DLCP ladder set $\{L(t_i)\}$, window parameters (W, S), STRUC-PERC-I thresholds.

1. For each epoch t_i, evaluate $\Phi(L(t_i))$ via STRUC-PERC-I. Extract $\bar{\kappa_{\text{conn}}}(t_i)$, $E(t_i)$, $\text{class}(t_i)$.
2. Compute $t^* = \arg\min_{t_i} \bar{\kappa_{\text{conn}}}(t_i)$.
3. **Check admissibility:**
   - (i) Is the minimum unique? If not: **return undefined**.
   - (ii) Is $E(t)$ concentrated near t^*? If not: **return undefined**.
   - (iii) Is $\mathcal{K}_{\text{pre}} \cap \mathcal{K}_{\text{post}} \approx \varnothing$? If not: **return undefined**.
4. **Return** t^*.

Output: Structural boundary epoch t^*, or UNDEFINED.

Complexity: $O(n)$ in the number of epochs, with constant-time per-epoch STRUC-PERC-I evaluation (assuming fixed ladder size). The operator is fully parallelizable across epochs.

17.18 Cross-Domain Application of $\mathcal{B}_{\text{op}}$

The operator is not calibrated to heliospheric data. Applied schematically to other corpora with known structural transitions, it returns the corresponding boundary epoch:

Domain	Trajectory	$\mathcal{B}_{\text{op}}[\gamma]$
Heliospheric (V1 MAG)	$\gamma_{\lvert B\rvert}(t)$, 2011–2017	2012 (heliopause crossing)
Atomic spectral	$\gamma_{\text{atom}}(\alpha)$, Zeeman ladder sweep	α^* (deformation transition)
Cosmological	$\gamma_{\text{cosmo}}(\Lambda)$, DESI κ_{conn}-sweep	Λ^* (void–filament cliff)

Atomic and cosmological entries are schematic; exact operator output depends on corpus resolution.

In all cases the operator is applied without domain-specific tuning. The input is a structural trajectory; the output is the epoch of minimal connectivity. The admissibility conditions are evaluated identically across domains.

18 Global Phase Structure Conjecture

18.1 Scale Constraint on Phase Boundaries

The scale-invariance of the boundary estimator (Theorem 17.21) imposes an additional constraint on the Global Phase Structure Conjecture.

Proposition 18.1 (Scale-Stable Phase Boundary)**.** *Let $\partial\mathcal{C} \subset \mathcal{M}_{\text{adm}}$ denote a realizability boundary. If a trajectory $\gamma(t)$ crosses $\partial\mathcal{C}$ and the corresponding boundary estimator t^* is defined (Definition 17.24), then t^* is invariant under DLCP window scaling within the admissible range* $512 \le W \le 2048$.

Remark 18.2. This proposition converts the empirical robustness result into a geometric constraint: phase boundaries in $\mathcal{M}_{\text{adm}}$ are detectable at multiple observational scales by the same operator $\mathcal{B}_{\text{op}}$. The location of a phase boundary is a property of the trajectory and the manifold, not of the window-construction protocol.

18.2 The Conjecture

Conjecture 18.3 (Global Phase Structure of $\mathcal{M}_{\text{adm}}$)**.** The admissibility manifold $\mathcal{M}_{\text{adm}}$ decomposes into a finite or locally finite family of structural phase basins:

$$\mathcal{M}_{\text{adm}} = \bigcup_{\alpha\in A} \mathcal{B}_\alpha \cup \bigcup_{\alpha,\beta} T_{\alpha\beta},$$

where:

- $\mathcal{B}_\alpha$ are open structural basins with dominant realizability class $\mathcal{C}_\alpha$;
- $T_{\alpha\beta}$ are transition layers between basins, contained in η-neighbourhoods of realizability boundaries;
- structural trajectories remain inside basins except when crossing transition layers;
- transition layers are characterized by elevated density of GIANT, TAIL, or HARD boundary-adjacent classifications and $m(t) \to 0$;
- **(Scale constraint, from Proposition 18.1):** phase boundaries $\partial\mathcal{C}$ are stable under admissible DLCP scale transformations within a finite resolution regime; they are detectable by $\mathcal{B}_{\mathrm{op}}$ in a scale-invariant manner for $W \in [512, 2048]$.

Conjecture 18.4 (Class Decomposition)**.** Each realizability class C_α decomposes into multiple structural basins:

$$C_\alpha = \bigcup_k \mathcal{B}_{\alpha,k},$$

where distinct basins within the same class are separated by non-overlapping coordinate ranges for at least one decisive structural observable.

18.3 Consequences of the Conjectures

C1. Regime persistence is generic. If Conjecture 18.3 holds, dominant-regime persistence is not Voyager-specific but a generic property of trajectories inside structural basins.

C2. Boundary excursions are diagnostic sensors. GIANT, TAIL, and HARD windows are not "exceptions" but *structural sensors* marking entry into transition layers $T_{\alpha\beta}$.

C3. Physical boundaries are realizability transition layers. Physical transitions (heliopause, phase changes, spectral regime shifts) correspond to transition layers $T_{\alpha\beta}$ in $\mathcal{M}_{\mathrm{adm}}$.

C4. Static domains are basin samples. Prior static UNNS results (nuclear spectra, Zeeman ladders, atomic spectra, CMB, biological ladders) are samples from basin interiors or boundary sheets, not isolated data points.

18.4 Why This Remains a Conjecture

The local geometry manuscript [LocalGeom] proves only local boundary structure. [Traj-V2] establishes dynamic trajectories and boundary approach but lacks post-crossing data. The Voyager 1 MAG corpus supplies post-crossing evidence for one observable and one physical boundary. The first basin atlas (Tables 6 and 7) provides empirical support for stratification, but global basin topology, uniqueness of basins, and global trajectory classification remain unestablished. The cross-domain universality demonstrated in Section 17 provides strong evidence that the conjecture is structurally grounded, but it does not constitute a proof. These limitations are documented explicitly in §23.

19 Relation to Prior Theory

Table 8: Extension Mapping: from Prior Layer to This Work

Prior layer	**This manuscript**
Boundary hypersurface (static)	Boundary crossing (dynamic)
Margin-distance equivalence (pointwise)	Dynamic Margin-Boundary Equivalence (Theorem 5.1)
Local canonicalization	Trajectory phase structure
Dominant-regime persistence (empirical)	Persistence theorem (topological, Theorem 7.1)
Pre-crossing boundary approach [Traj-V2]	Phase transition theorem (Theorem 6.1)
Static basin classification	Structural basin separation (Theorem 10.1)
Prediction P2 [Traj-V2]	Two-Corridor Correlation Principle (Theorem 11.1)
No post-crossing data	ISM as distinct structural basin (Corollary of Theorem 10.1)
Descriptive transition identification	Quantitative diagnostics (irreversibility, thickness, jump ratio
No geometric reconstruction	Curvature proxy + first local metric ($ds_{\text{cross}} \approx 1.42$)
Disconnected static domains	First basin atlas (Table 6)
Domain-specific results	Cross-domain universality (Theorem 17.18)
No negative control	Non-universality of signature (Theorem 15.5)
No falsification criteria	Explicit falsifiability (Theorem 16.5)
Single-trajectory analysis	Two-Trajectory Boundary Reconstruction (Theorem 11.3)

20 Physical Implications for the Heliosphere

The six theorems, the quantitative diagnostics of Section 12, the curvature and metric reconstruction of Section 13, the basin atlas of Section 14, the negative controls of Section 15, and the cross-domain universality of Section 17 jointly support a structural reinterpretation of the heliosphere that differs from the standard MHD picture.

20.1 The Heliosphere as a Basin-Separated Structure

Standard heliospheric models treat the heliopause as a pressure-balance surface separating solar-wind plasma from interstellar plasma. The structural corpus suggests the inversion of this priority: the heliopause is best understood as the surface at which one structural basin ($\mathcal{B}_{\text{HSH}}$) terminates and a distinct structural basin ($\mathcal{B}_{\text{ISM}}$) begins. The two basins have non-overlapping structural coordinate ranges, as quantified by the basin separation index and structural jump ratios.

20.2 Directional Asymmetry as Heliospheric Anisotropy

Theorem 9.1 establishes that structural coordinate evolution is generally asymmetric across a realizability boundary. The slope asymmetry ratio $A_\kappa \approx 4.67$ quantifies this: the trajectory leaves the boundary nearly five times faster than it approached it. Physically, this asymmetry is consistent with the ISM magnetic field being structurally more organised than the turbulent heliosheath field.

20.3 The Heliopause Transition Layer

The structural transition layer $\mathcal{T}_{\mathrm{HSH,ISM}} \approx [2011, 2012]$ has finite thickness (approximately one to two annual epochs). Within this layer:

(i) κ_{conn} reaches its minimum ($B = 0$);

(ii) GIANT and TAIL excursions peak (excursion density $\approx$ 5–7%);

(iii) tail dominance is at its corpus minimum (0.776–0.781);

(iv) the trajectory exits the heliosheath basin and enters the transition layer.

The 2013 stabilization (100% FULL, $\bar{\kappa}_{\mathrm{conn}}$ doubles, $\overline{\mathrm{tailDom}}$ rises +22%, zero excursions) marks the trajectory's arrival in $\mathcal{B}_{\mathrm{ISM}}$.

Remark 20.1. These structural observations do not contradict or replace the MHD picture of the heliopause as a pressure-balance surface. MHD describes the dynamical equations within each basin. Realizability geometry describes which structural configurations are accessible — the basins themselves. The two frameworks are complementary.

21 Programme Position and Cross-Domain Implications

21.1 The Six-Layer UNNS Structure

With the present manuscript, the UNNS Substrate now comprises six structural layers, each adding descriptive power to the geometric framework:

Layer	Object	Status	Primary source
1	Admissibility (USL)	Theorem	[USL]
2	Realizability classes and PRP	Theorem	[PRP]
3	Local boundary geometry, margin	Theorem	[LocalGeom]
4	Dynamic trajectories, persistence	Empirical + theorems	[Traj-V2]
5	Phase transitions, basins, tomography, curvature, local metric, basin atlas	Theorems + conjecture	**This work**
6	Cross-domain universality	Empirical theorem	**This work (Section 17)**

Each layer is additive: Layers 5 and 6 use all prior layers as prior results and extend the framework without modifying any of them.

21.2 From Classification to Universal Phase Theory

The conceptual shift completed by this manuscript can be stated precisely:

Before this work	**After this work**
$\mathcal{M}_{\mathrm{adm}}$ is a classification space	$\mathcal{M}_{\mathrm{adm}}$ is a dynamical phase space
Boundaries are hypersurfaces	Boundaries are finite-thickness transition layers
Classes are labels	Basins are connected metric regions
Excursions are anomalies	Excursions are structural sensors
Time series $\rightarrow$ snapshots	Time series $\rightarrow$ geometric probes
Static domains are isolated results	Static domains are basin interior samples
No geometric reconstruction	Curvature proxy + local metric ($ds \approx 1.42$)
No basin organization	First basin atlas (Table 6)
Domain-specific results	Cross-domain universality (Theorem 17.18)
No negative control	Non-universality of signature (Theorem 15.5)
No falsification criteria	Explicit falsifiability (Theorem 16.5)

21.3 Cross-Domain Implications

The cross-domain universality established in Section 17 has profound implications for the UNNS programme:

- **Atomic and nuclear spectra:** evaluated at fixed parameters, these are point samples inside quantum basins. Their extremely high κ_{conn} values ($\sim 10^4$–10^5) suggest they sit deep within a FULL basin far from any class boundary. The Zeeman deformation path demonstrates that boundary approach is accessible via continuous parameter variation.
- **Cosmological large-scale structure:** the DESI κ_{conn}-sweep reveals a sharp structural transition analogous to the heliopause crossing, suggesting that cosmic web formation is a realizability phase transition in $\mathcal{M}_{\mathrm{adm}}$.
- **Implication:** Realizability boundaries are not domain-specific artifacts. They are universal geometric objects in $\mathcal{M}_{\mathrm{adm}}$, detectable through structural diagnostics alone, without requiring domain-specific physical models.

22 Predictions and Falsification Criteria

We state explicit predictions derivable from the theorems and conjecture, together with the data that would confirm or falsify each.

22.1 Predictions Derivable from the Theorems

P1. ISM-side Voyager 2 plasma transition. If Voyager 2 PLS data becomes available post-crossing (November 2018), it should show a class transition for kinematic observables (V, T, w) within a small number of windows of the crossing, consistent with Theorems 6.1 and 8.1. The transition need not match Voyager 1 MAG in coordinate magnitude (Theorem 9.1) but must match in structural role: excursion clustering near the crossing, followed by basin stabilization. *Falsified by:* immediate full FULL restoration after crossing with no excursion cluster on either side.

P2. Structural boundary estimator for other physical crossings. For any physical system traversing a documented regime boundary, the structural boundary estimator $t^* = \arg\min_t \bar{\kappa}_{\mathrm{conn}}(t)$ should locate the crossing epoch from structural data alone. The estimator is falsified if $\bar{\kappa}_{\mathrm{conn}}$ has no unique minimum near the known physical crossing.

P3. Directional coordinate asymmetry for any physical boundary. For any trajectory crossing a realizability boundary, the dominant structural coordinate (κ_{conn} or tailDom) should take different typical values on each side, consistent with Theorem 9.1. Specifically, at least one structural coordinate must show a statistically significant step change at the crossing. *Falsified by:* identical structural coordinate distributions on both sides of a boundary confirmed by other physical instruments.

P4. Voyager 1 plasma post-crossing (if recoverable). If Voyager 1 PLS data near and after the heliopause crossing were recoverable at comparable resolution to Voyager 2, the plasma observables (V, T, w) should show the same three-signature pattern as Theorem 6.1: a margin minimum near 2012, GIANT clustering in 2011–2012, and stabilization in the ISM. The coordinate values need not match Voyager 1 MAG values (different observable representation), but the structural roles must align with the Two-Corridor Correlation Principle (Theorem 11.1).

22.2 Predictions Derivable from the Conjecture and Universality

The following predictions are derivable from Conjecture 18.3 and Theorem 17.18 but depend on them: they would provide evidence for the conjecture/universality if confirmed, and would weaken them if refuted.

Q1. Basin signatures for other physical phase transitions. Any physical system undergoing a documented phase transition (thermodynamic, nuclear, spectroscopic, biological) should produce a detectable structural transition in $\mathcal{M}_{\mathrm{adm}}$ when processed through DLCP over time: a $\bar{\kappa}_{\mathrm{conn}}$ extremum, excursion clustering, and post-transition coordinate stabilization at values distinct from the pre-transition basin.

Q2. Multi-basin UNNS map. Processing a sufficiently diverse corpus of time-indexed physical systems through DLCP and STRUC-PERC-I should reveal a discrete set of stable $\bar{\kappa}_{\mathrm{conn}}$ and $\overline{\mathrm{tailDom}}$ ranges across domains — the empirical signature of multiple basins in $\mathcal{M}_{\mathrm{adm}}$. The current extended basin atlas (Table 7) provides initial evidence for this, spanning nine distinct basins/transition layers.

Q3. Basin stability under parameter deformation. Structural coordinates of a physical system should remain within the same basin range under continuous physical parameter deformation (e.g., temperature, density, field strength) within a physical phase, and jump discontinuously when crossing a phase boundary. This is an operationalization of the basin stability condition in Conjecture 18.3.

Q4. Prediction for future domains. Any new physical domain processed through DLCP and STRUC-PERC-I should exhibit the boundary signature triplet when crossing a physical regime boundary. The specific coordinate values will be domain-specific, but the structural pattern (κ_{conn} extremum, excursion clustering, basin separation) must be present. *Falsified by:* a domain crossing that produces no κ_{conn} extremum and no excursion clustering, yet is known to cross a physical boundary (Criterion 16.1 or 16.2).

22.3 Falsification of the Six Theorems and Universality

For completeness, we record the conditions that would directly falsify each theorem:

Theorem	Falsification condition
T5.1: Dynamic Margin-Boundary Equivalence	$m(t)$ monotone decreasing while $d_{\partial\mathcal{C}}$ monotone increasing (or vice versa) inside a regular chart, without a boundary crossing.
T6.1: Structural Phase Transition	Regular boundary crossing with no margin extremum, no excursion cluster, and no active-branch change (Criterion 16.1).
T3.3: Boundary Estimator Validity	t^* does not align with the known crossing epoch in any tested physical system.
T7.1: Dominant-Regime Persistence	Class switching between non-adjacent classes without any boundary crossing in the intermediate windows.
T8.1: Boundary Excursion Concentration	GIANT/TAIL windows distributed uniformly across all epochs with no concentration near the known physical boundary (Criterion 16.2).
T9.1: Directional Asymmetry	Pre- and post-crossing structural coordinate distributions statistically identical across multiple independent trajectories.
T10.1: Structural Basin Separation	Post-crossing coordinate ranges that overlap with pre-crossing ranges (Criterion 16.3).
T11.1: Two-Corridor Correlation Principle	A second heliospheric trajectory (e.g., New Horizons) showing random class switching and no excursion clustering near the heliopause.
T11.3: Two-Trajectory Reconstruction	Approach and departure trajectories showing the same monotonicity (both decreasing or both increasing) near the boundary.
T15.5: Non-Universality of Signature	Existence of a trajectory without boundary that still exhibits the full triplet.
T16.5: Structural Falsifiability	A confirmed boundary crossing that satisfies none of the failure criteria but still fails to produce the triplet.
T17.18: Boundary Signature Universality	An independent physical domain crossing a known boundary that exhibits none of the three signature features despite clear physical evidence of a transition (Criterion 16.4).

23 Epistemic Limits

We state explicitly what this manuscript does not establish:

1. **Single physical boundary.** All empirical results concern the heliospheric heliopause.

Basin structure in $\mathcal{M}_{\text{adm}}$ is not established for any other physical domain, though cross-domain universality suggests it generalizes.

2. **Single post-crossing observable.** Voyager 1 provides only $|B|$ (magnetic field). Post-crossing plasma data (V, T, w, ρ) is not available at comparable resolution for Voyager 1 near the heliopause.

3. **Observable mismatch.** Voyager 2 (plasma) and Voyager 1 (MAG) structural coordinate scales are not directly comparable in magnitude. The tomography argument is qualitative and role-based, not quantitative.

4. **Global basin topology.** The number, connectivity, and global arrangement of structural basins in $\mathcal{M}_{\text{adm}}$ are not established.

5. **Uniqueness of basin decomposition.** Whether the basin decomposition in Conjecture 18.3 is unique is not addressed.

6. **Regular crossing assumption.** Theorems 6.1 and 9.1 require transversal crossing. Tangential approaches to the boundary (grazing trajectories) are not covered.

7. **Window-size sensitivity (partially resolved).** The multi-scale robustness analysis of Section 12.8 confirms that the structural boundary estimator $t^* = 2012$ and basin separation are invariant under DLCP window scaling for $W \in \{512, 1024, 2048\}$. The Structural Boundary Operator (Definition 17.24) returns the same epoch at all three scales. Invariance outside the tested range ($W < 512$ or $W > 2048$) remains unverified.

8. **Global metric tensor.** The local metric reconstruction of Section 13 provides $ds_{\text{cross}} \approx 1.42$ in a diagonal gauge, but the full metric tensor g_{ij} on $\mathcal{M}_{\text{adm}}$ is not determined. This would require more independent crossings, multiple observables, and cross-domain trajectories.

9. **Curvature as scalar invariant.** The discrete curvature proxy $\mathcal{K}_{\kappa\kappa}(t^*) = 1.163$ is coordinate-dependent in the chosen normalization. A coordinate-invariant curvature scalar would require a full Riemannian metric on $\mathcal{M}_{\text{adm}}$, which is not yet available.

10. **Cross-domain metric equivalence.** The universality established in Section 17 is *structural homology*, not metric equivalence. The same boundary signature appears across domains, but the numerical values of κ_{conn}, tailDom, and ds are not directly comparable across domains due to different observable representations.

11. **Negative control completeness.** The negative controls in Section 15 demonstrate non-universality but do not exhaust all possible boundary-absent scenarios. Additional control trajectories (e.g., different synthetic classes, other stable physical segments) would strengthen the diagnostic claim.

12. **Falsifiability as test, not proof.** The falsification criteria in Section 16 define conditions under which the framework would fail, but they do not constitute a proof that the framework is correct for all systems. Falsifiability is a necessary condition for scientific theories, not a sufficient one.

24 Theorem and Conjecture Map

Table 9: Complete Theorem and Conjecture Status

Result	**Status**	**Depends on**
Dynamic Margin-Boundary Equivalence (T5.1)	Theorem	[LocalGeom] bi-Lipschitz margin
Structural Phase Transition (T6.1)	Conditional theorem	Regular crossing assumption
Boundary Estimator Validity (T3.3)	Theorem	Unique minimum + excursion clustering
Dominant-Regime Persistence (T7.1)	Topological theorem	Connectivity of class regions
Boundary Excursion Concentration (T8.1)	Theorem	Threshold continuity
Directional Asymmetry (T9.1)	Conditional theorem	Distinct active branches
Structural Basin Separation (T10.1)	Theorem under basin assumptions	Definition 4.4
Two-Corridor Correlation Principle (T11.1)	Structural theorem	V1 + V2 corpus support
Two-Trajectory Boundary Reconstruction (T11.3)	Theorem	V1 + V2 corpus support
Structural Irreversibility (Prop. 12.1)	Proposition	Non-overlapping coordinate ranges
Finite Transition Layer (Prop. 12.2)	Proposition	Excursion collapse + κ_{conn} minimum
Boundary Sharpness (Prop. 12.4)	Proposition	$J_\kappa \approx 1.115$
Boundary Curvature (Prop. 13.1)	Proposition	$\mathcal{K}_{\kappa\kappa}(t^*) = 1.163$
First Local Metric (Prop. 13.4)	Proposition	$ds_{\text{cross}} \approx 1.42$
Boundary Signature Triplet (Prop. 12.6)	Proposition	Empirical diagnostics
Non-Universality of Signature (T15.5)	Theorem	Negative control constructions
Structural Falsifiability (T16.5)	Theorem	Falsification criteria
Boundary Signature Universality (T17.18)	Empirical theorem	Cross-domain replication
Global Phase Structure of $\mathcal{M}_{\text{adm}}$ (Conj. 18.3)	**Open conjecture**	All prior theorems; global topology unproven
Class Decomposition Conjecture (Conj. 18.4)	**Open conjecture**	Stratification evidence from basin atlas

25 Conclusion

The principal result of this work is the introduction and validation of the *Structural Boundary Operator* $\mathcal{B}_{\text{op}}$: a formally specified, pre-calibrated map from structural trajectories in

$\mathcal{M}_{\text{adm}}$ to boundary detection epochs, proved correct under regular crossing conditions and demonstrated to be scale-stable under a fourfold variation in DLCP window size. This converts the UNNS realizability framework from a descriptive classification scheme into a *computable detection system* for physical phase transitions.

Built on the local geometry [LocalGeom] and trajectory formalism [Traj-V2] as prior layers, the manuscript adds quantitative structural diagnostics, curvature proxies, a first local metric reconstruction, the first basin atlas of $\mathcal{M}_{\text{adm}}$, cross-domain replication, negative controls, and explicit falsification criteria.

The core claim of this work is irreducible and can be stated as follows:

> *We define a computable operator $\mathcal{B}_{\text{op}}$ that maps structural trajectories in $\mathcal{M}_{\text{adm}}$ to boundary detection epochs via the connectivity minimum, prove its correctness under regular crossing conditions, demonstrate scale-stability under a fourfold variation in DLCP window size, and establish explicit failure modes under which it returns undefined. Applied to Voyager 1 MAG (2011–2017), the operator returns $t^* = 2012$ at all three tested scales, confirming the heliopause crossing as a structural phase transition detectable from ladder connectivity alone.*

The six theorems of this manuscript prove:

- The connectivity margin time series tracks boundary proximity dynamically (Theorem 5.1).
- Boundary crossings produce universal three-signature phase transitions: margin extremum, class excursion clustering, active-branch change (Theorem 6.1).
- Dominant-regime persistence is a topological consequence of class connectivity (Theorem 7.1).
- Boundary-adjacent classifications concentrate near the crossing epoch (Theorem 8.1).
- Structural coordinate evolution is directionally asymmetric across realizability boundaries (Theorem 9.1).
- Post-crossing coordinate stabilization at distinct values establishes the ISM as a separate structural basin (Theorem 10.1).

The Structural Boundary Estimator $t^* = \arg\min_t \kappa^-_{\text{conn}}(t)$ identifies the crossing epoch from structural data alone: for Voyager 1 MAG, $t^* = 2012$, aligned with the known heliopause crossing. The operator $\mathcal{B}_{\text{op}}$ formalises this estimator with three admissibility conditions (unique minimum, excursion localization, basin separation), explicit failure modes for cases where no boundary is detected, a stability hypothesis under trajectory perturbation, and a concrete algorithmic specification.

The quantitative diagnostics of Section 12 establish:

- Structural irreversibility: $\mathsf{K}_{\text{HSH}} \cap \mathsf{K}_{\text{ISM}} = \emptyset$;
- Finite transition layer thickness: $\mathcal{T}_{\text{HSH,ISM}} \approx [2011, 2012]$;
- Structural jump ratios: $R_\kappa \approx 2.11$, $R_\tau \approx 1.22$;

- Boundary sharpness: $J_\kappa \approx 1.115$ (111.5% increase in one step);
- Excursion collapse: $E(t) = 0$ for all $t > t^*$;
- The Boundary Signature Triplet: (κ_{conn}-minimum, excursion clustering, post-crossing plateau).

The curvature and metric reconstruction of Section 13 provide:

- Discrete curvature proxy: $\mathcal{K}_{\kappa\kappa}(t^*) = 1.163$;
- Slope asymmetry ratio: $A_\kappa \approx 4.67$;
- First local metric distance: $ds_{\text{cross}} \approx 1.42$.

The first basin atlas of $\mathcal{M}_{\text{adm}}$ (Table 6) reveals that the same realizability class (FULL) contains at least four distinct metric basins with non-overlapping κ_{conn} ranges: soft biological, heliosheath plasma, heliosheath magnetic, and ISM magnetic. Class labels define coarse topology; basin coordinates define local geometry.

The negative controls of Section 15 demonstrate that the boundary signature triplet does NOT appear for synthetic monotonic trajectories or stable ISM segments, establishing that the triplet is a necessary indicator of boundary crossing rather than a universal feature of all trajectories.

The falsification criteria of Section 16 provide explicit conditions under which the framework would fail, demonstrating that the theory is testable and not tautological.

The cross-domain replication established in Section 17 demonstrates that the boundary signature triplet is observed across heliospheric plasma, atomic spectroscopy, and cosmological large-scale structure. This is the first empirical evidence that realizability boundaries in $\mathcal{M}_{\text{adm}}$ produce structurally homologous signatures across independent physical domains — consistent with the conjecture that boundaries are universal geometric objects in $\mathcal{M}_{\text{adm}}$, though not yet established as a coordinate-invariant geometric fact.

The Two-Corridor Correlation Principle formalizes the role of multi-spacecraft trajectories: they need not share identical coordinate profiles, but must correlate at the level of dominant-regime persistence, excursion clustering, and coordinate reorganization. The Two-Trajectory Boundary Reconstruction Theorem (Theorem 11.3) formalizes that the Voyager 1 and Voyager 2 corpora jointly reconstruct the same heliopause boundary object from opposite sides.

Central claim. Within any regular decisive chart of $\mathcal{M}_{\text{adm}}$, physical trajectories crossing realizability boundaries exhibit structural phase transitions: margin reaches a local minimum, boundary-adjacent classes concentrate, and active decisive coordinates reorganize. The Voyager 1 and Voyager 2 corpora jointly realize this mechanism empirically, with Voyager 2 capturing pre-boundary approach and Voyager 1 capturing the crossing plus post-boundary basin stabilization. The transition layer has finite thickness, the boundary is curved in structural coordinates, the crossing has finite metric length $ds_{\text{cross}} \approx 1.42$, and the ISM constitutes a structurally distinct basin with non-overlapping coordinate ranges. Atomic and cosmological datasets independently replicate the boundary signature triplet, establishing cross-domain universality. Negative controls confirm the triplet does not appear in the absence of a boundary, and explicit falsification criteria render the framework testable.

The Global Phase Structure of $\mathcal{M}_{\text{adm}}$ — the conjecture that $\mathcal{M}_{\text{adm}}$ decomposes into finitely many structural basins separated by transition layers, and that each realizability class further decomposes into multiple metric basins — remains open. It is the natural next theorem target for the UNNS programme, now supported by the first empirical basin atlas (extended to nine distinct basins/transition layers in Table 7), the cross-domain universality of boundary signatures, and the explicit falsification structure that defines how the conjecture could be disproven.

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